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H. Xiao

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On scalar-, vector- and second-order tensor-valued anisotropic functions of vectors and second-order tensors relative to all kinds of subgroups of the transverse isotropy group $C_{\infty h}$

By H. Xiao

Institute of Mechanics I, Ruhr-University Bochum, D-44780 Bochum, Germany and Department of Mathematics, College of Mathematical Science, Peking University, Beijing 100871, People's Republic of China

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Contents

Scalar-, vector- and second-order tensor-valued functions of any finite number of vector- and second-order tensor variables are considered, which are required to possess invariance or form-invariance under the action of any given subgroup of the transverse isotropy group $C_{\infty h}$. *Irreducible functional bases* and *irreducible generating sets* are presented to determine general reduced forms of the anisotropic functions just indicated. The results given are derived in the sense of non-polynomial representation.

Keywords: vectors and tensors; anisotropic functions; constitutive equations; material symmetry groups; invariance and form-invariance; irreducible non-polynomial representations

1. Introduction

In continuum physics, scalar-, vector- and second-order tensor-valued functions of scalar-, vector- and second-order tensor† variables serve as mathematical models of macroscopic physical behaviours of materials, such as elasticity, elastoplasticity, viscoelasticity, creep, damage, photoelasticity, piezoelectricity, electromagnetic elasticity, etc. Of them, the variables may be mass density, temperature, work-hardening parameter, the gradient of temperature, electric and magnetic field vectors, strain tensors, strain-rate tensors, and some internal variables characterizing internal states of materials, etc., while the values may be energy, entropy, heat-flux vector, stress tensors, stress-rate tensors, and rates of internal variables, etc. The principles of material objectivity and material symmetry require that such tensor functions modelling material behaviours, commonly known as *material constitutive relations or equations*, obey a combined form-invariance restriction under the material symmetry group. Specifically, let $f(\boldsymbol{u}_i, \boldsymbol{W}_\sigma, \boldsymbol{A}_L)$, $\boldsymbol{h}(\boldsymbol{u}_i, \boldsymbol{W}_\sigma, \boldsymbol{A}_L)$ and $\boldsymbol{F}(\boldsymbol{u}_i, \boldsymbol{W}_\sigma, \boldsymbol{A}_L)$ be, respectively, scalar-, vector- and second-order tensor-valued functions of the a vector variables $\boldsymbol{u}_i$, the b skewsymmetric second-order tensor variables $\boldsymbol{W}_\sigma$ and the c symmetric second-order tensor variables $\boldsymbol{A}_L$, where $i = 1, \ldots, a$, $\sigma = 1, \ldots, b$ and $L = 1, \ldots, c$. Moreover, let g be a subgroup of the full orthogonal group Orth, which may serve as a material symmetry group for solid materials. The tensor functions f, $\boldsymbol{h}$ and $\boldsymbol{F}$ are invariant (for f) or form-invariant (for $\boldsymbol{h}$ and $\boldsymbol{F}$) relative to or under the group g, respectively, if

$$f(\boldsymbol{Q}\boldsymbol{u}_i, \boldsymbol{Q}\boldsymbol{W}_\sigma\boldsymbol{Q}^{\mathrm{T}}, \boldsymbol{Q}\boldsymbol{A}_L\boldsymbol{Q}^{\mathrm{T}}) = f(\boldsymbol{u}_i, \boldsymbol{W}_\sigma, \boldsymbol{A}_L),$$
$$\boldsymbol{h}(\boldsymbol{Q}\boldsymbol{u}_i, \boldsymbol{Q}\boldsymbol{W}_\sigma\boldsymbol{Q}^{\mathrm{T}}, \boldsymbol{Q}\boldsymbol{A}_L\boldsymbol{Q}^{\mathrm{T}}) = \boldsymbol{Q}\boldsymbol{h}(\boldsymbol{u}_i, \boldsymbol{W}_\sigma, \boldsymbol{A}_L),$$
$$\boldsymbol{F}(\boldsymbol{Q}\boldsymbol{u}_i, \boldsymbol{Q}\boldsymbol{W}_\sigma\boldsymbol{Q}^{\mathrm{T}}, \boldsymbol{Q}\boldsymbol{A}_L\boldsymbol{Q}^{\mathrm{T}}) = \boldsymbol{Q}\boldsymbol{F}(\boldsymbol{u}_i, \boldsymbol{W}_\sigma, \boldsymbol{A}_L)^{\mathrm{T}}\boldsymbol{Q},$$

for every $\boldsymbol{Q} \in g$. General reduced forms of the tensor functions f, $\boldsymbol{h}$ and $\boldsymbol{F}$ under the above invariance restrictions are called *representations* for f, $\boldsymbol{h}$ and $\boldsymbol{F}$ under the group g. It has been known (see Pipkin & Wineman 1963; Wineman & Pipkin 1964; Spencer 1971; Zheng & Boehler 1994) that finding representations for the tensor functions f, $\boldsymbol{h}$ and $\boldsymbol{F}$ under a given group g is equivalent to determining *functional bases* (for f) or *generating sets* (for $\boldsymbol{h}$ and $\boldsymbol{F}$) under the group g. Moreover, both functional bases and generating sets to be used are further required to be *irreducible* (see Spencer 1971; see also Xiao (1996*a*) and below) in order to arrive at compact representations.

Subgroups of the full orthogonal group Orth include the five classes of cubic crystal groups, the two classes of icosahedral groups, the five classes of transverse isotropy groups, and their denumerably infinitely many classes of subgroups (see Vainshtein 1994; see also Zheng & Boehler 1994). Of them, the 32 classes of crystal groups, the five classes of transverse isotropy groups and the full and proper orthogonal groups Orth and Orth$^+$ are related to commonly met solid materials (see, for example, Truesdell & Noll 1965; Spencer 1971), while the others are associated with quasi-crystalline solids and texture materials, etc. (see Vainshtein 1994). In the past decades, many efforts have been made to find representations for various kinds of tensor functions under the aforementioned orthogonal subgroups, and many important results obtained. Here, I will not give the large number of related references. For

† Throughout, vector and tensor mean three-dimensional vector and tensor.

further details, refer to Truesdell & Noll (1965), Spencer (1971), Eringen & Maugin (1989), Kiral & Eringen (1990), Rychlewski (1991), Smith (1994) and the recent reviews by Betten (1991), Rychlewski & Zhang (1991) and Zheng (1994), as well as the references therein. Although many results in many cases, mainly in the sense of polynomial representation, are now available, the general aspect of representation problems for anisotropic functions, mainly in the sense of non-polynomial representation, remains open, except for some particular cases such as isotropic, orthotropic and transversely isotropic functions, etc. (see, for example, Adkins 1958, 1960; Wang 1970; Smith 1971, 1982; Boehler 1977; Pennisi & Trovato 1987; Zheng 1993*a*; Jemioło & Telega 1997). Based upon the isotropic extension method for anisotropic functions, initiated by Lokhin & Sedov (1963), Boehler (1979, 1987) and Liu (1982) (see also Rychlewski 1991; Rychlewski & Zhang 1991; Zheng & Spencer 1993) and recently developed by this author (see Xiao & Guo 1993; Xiao 1995*a*, 1996*a*), as well as the recent results given in Xiao (1995*b*, 1996*b*–*d*, 1998*a*, 1999), we shall derive general irreducible representations for scalar-, vector- and second-order tensor-valued anisotropic functions relative to various kinds of orthogonal subgroups in a series of works. In this paper, we shall investigate anisotropic functions relative to all kinds of subgroups of the transverse isotropy group $C_{\infty h}$ (see the related references listed before and Smith & Rivlin (1958, 1964), Smith (1962, 1982), Spencer (1971), Smith & Kiral (1969), Kiral & Smith (1974), Zheng (1993*a*), among others, for some particular results of this aspect; in particular, see Zheng (1993*b*) for the two-dimensional counterpart).

In the remaining part of this introduction, I state some facts that will be used. Throughout, V, Skw and Sym are used to represent the vector space, the skewsymmetric and symmetric second-order tensor spaces, respectively. The notation $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{r}$, etc., $\boldsymbol{W}$, $\boldsymbol{H}$, etc., $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, etc., is used to denote vectors, skewsymmetric second-order tensors and symmetric second-order tensors, respectively. On the other hand, $g(X)$ is the symmetry group of the set X of vectors and tensors, and for any subgroup $g \subset \text{Orth}$, $M(g)$ the g-subspace of the space $M \in \{V, \text{Skw}, \text{Sym}\}$. The former is composed of all orthogonal tensors preserving X, while the latter is formed by all tensors each of which is invariant under the group $g \subset \text{Orth}$ (see Xiao (1996*b*) for details).

A finite set of scalar-valued functions that are invariant under group $g \subset \text{Orth}$, $\{f_1, \ldots, f_r\}$, is a functional basis under g if each scalar-valued function f that is invariant under g is expressible as a single-valued function of the just-stated invariants. On the other hand, a finite set of vector-valued (resp. second-order tensor-valued) functions that are form-invariant under the group $g \subset \text{Orth}$, $\{\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_s\}$, is a generating set under g if each vector-valued (resp. second-order tensor-valued) function $\boldsymbol{\psi}$ is expressible as a linear combination of the just-stated set whose coefficients are invariants under g. Furthermore, a functional basis (resp. a generating set) under the group $g \subset \text{Orth}$ is irreducible if none of its proper subsets is again a functional basis (resp. a generating set) under the group g. According to Xiao (1996*b*), a criterion for a generating set is as follows.

Criterion 1.1. *The vector-valued or skewsymmetric tensor-valued or symmetric tensor-valued functions,* $\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_r$, *that are form-invariant under the group* $g \subset \text{Orth}$ *form a generating set under* g *iff*

$$\text{rank}\{\boldsymbol{\psi}_1(\boldsymbol{u}_i, \boldsymbol{W}_\sigma, \boldsymbol{A}_L), \ldots, \boldsymbol{\psi}_r(\boldsymbol{u}_i, \boldsymbol{W}_\sigma, \boldsymbol{A}_L)\} \geqslant \dim M(g \cap g(\boldsymbol{u}_i, \boldsymbol{W}_\sigma, \boldsymbol{A}_L)) \quad (1.1)$$

for all $(\boldsymbol{u}_i, \boldsymbol{W}_\sigma, \boldsymbol{A}_L) \equiv (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_a; \boldsymbol{W}_1, \ldots, \boldsymbol{W}_b; \boldsymbol{A}_1, \ldots, \boldsymbol{A}_c) \in V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$, *where* $M = V$, Skw, Sym, *respectively, when* $\boldsymbol{\psi}_i$ *is vector-valued, skewsymmetric tensor-valued and symmetric tensor-valued, respectively.*

Throughout, the symbols $\operatorname{rank}\mathcal{S}$ and $\dim\tilde{\mathcal{S}}$, where $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are a set of vectors or tensors and a vector or tensor subspace, are used to designate the number of the linearly independent elements in the set $\mathcal{S}$ and the dimension of the subspace $\tilde{\mathcal{S}}$, respectively.

To apply the above criterion, the following facts concerning the subgroups $g \subset C_{\infty h}$ will be useful:

$$\dim V(g) = \begin{cases} 3, & g = C_1, \\ 2, & g = C_{1h}, \\ 1, & g \subset \mathrm{Orth}^+, g \neq C_1, \\ 0, & \text{otherwise;} \end{cases} \tag{1.2}$$

$$\dim \mathrm{Skw}(g) = \begin{cases} 3, & g = C_1, S_2, \\ 1, & \text{otherwise;} \end{cases} \tag{1.3}$$

$$\dim \mathrm{Sym}(g) = \begin{cases} 6, & g = C_1, S_2, \\ 4, & g = C_2, C_{1h}, C_{2h}, \\ 2, & \text{otherwise.} \end{cases} \tag{1.4}$$

On the other hand, a criterion for functional bases is as follows (see Pipkin & Wineman 1963; Wineman & Pipkin 1964; see also Xiao 1996*b*).

Criterion 1.2. *The invariants* $f_1, \ldots, f_r$ *under the group* $g \subset \mathrm{Orth}$ *form a functional basis under* g *iff for* $\bar{X}$, $X \in V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$,

$$f_1(\bar{X}) = f_1(X), \ldots, f_r(\bar{X}) = f_r(X) \Longrightarrow \exists \boldsymbol{Q} \in g : \bar{X} = X. \tag{1.5}$$

To check the irreducibility of a given functional basis, we shall use the following fact.

Fact 1.3. A functional basis I under the group $g \subset \mathrm{Orth}$ is irreducible iff for any given element $f_0 \in I$ there exist $X, X' \in V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$, which belong to two different g-orbits, such that

$$f_0(X) \neq f_0(X') \quad \text{and} \quad f(X) = f(X') \text{ for all } f \in I/\{f_0\}. \tag{1.6}$$

In fact, for any given element f_0, the proper subset $I/\{f_0\}$ cannot be a functional basis under g if (1.6) holds, or else, according to the aforementioned criterion for functional bases, X and X' must pertain to the same g-orbit (see (1.5)) and hence $f_0(X) = f_0(X')$, which contradicts $(1.6)_1$.

In addition, by means of the Schoenflies symbol we record the transverse isotropy groups $C_{\infty h}$ and C_∞, as well as all their finite subgroups as follows:

$$C_{\infty h}(\boldsymbol{n}) = \{\pm \boldsymbol{R}_{\boldsymbol{n}}^{\theta} \mid \theta \in R\}, \quad C_\infty(\boldsymbol{n}) = C_{\infty h}(\boldsymbol{n}) \cap \mathrm{Orth}^+,$$
$$C_1 = \{\boldsymbol{I}\}, \qquad S_2 = \{\pm \boldsymbol{I}\};$$
$$C_{1h}(\boldsymbol{n}) = \{\boldsymbol{I}, -\boldsymbol{R}_{\boldsymbol{n}}^{\pi}\}, \qquad C_2(\boldsymbol{n}) = \{\boldsymbol{I}, \boldsymbol{R}_{\boldsymbol{n}}^{\pi}\}, \qquad C_{2h}(\boldsymbol{n}) = \{\pm \boldsymbol{I}, \pm \boldsymbol{R}_{\boldsymbol{n}}^{\pi}\};$$
$$S_{4m+2}(\boldsymbol{n}) = \{\pm \boldsymbol{R}_{\boldsymbol{n}}^{2k\pi/2m+1} \mid k = 1, \ldots, 2m+1\},$$
$$C_{2m+2h}(\boldsymbol{n}) = \{\pm \boldsymbol{R}_{\boldsymbol{n}}^{2k\pi/2m+2} \mid k = 1, \ldots, 2m+2\},$$

$$C_{2m+1}(\boldsymbol{n}) = S_{4m+2}(\boldsymbol{n}) \cap \mathrm{Orth}^+, \qquad C_{2m+2}(\boldsymbol{n}) = C_{2m+2h}(\boldsymbol{n}) \cap \mathrm{Orth}^+;$$
$$S_{4m}(\boldsymbol{n}) = \{(-\boldsymbol{R}_{\boldsymbol{n}}^{\pi/2m})^k \mid k = 1, \ldots, 4m\},$$
$$C_{2m+1h}(\boldsymbol{n}) = \{(-\boldsymbol{R}_{\boldsymbol{n}}^{\pi/2m+1})^k \mid k = 1, \ldots, 4m+2\}.$$

Here and hereafter, $\boldsymbol{n}$ is a given unit vector, $\boldsymbol{I}$ is the identity tensor and Orth^+ is the proper orthogonal group. Henceforth, we shall omit the defining vector $\boldsymbol{n}$ when citing the above groups, if no confusion arises.

Finally, for $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{r} \in V$ and $\boldsymbol{A} \in \mathrm{Sym}$, we shall use the following notation:

$$\left.\begin{aligned} \mathring{\boldsymbol{u}} &= \boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}, \\ \mathring{\boldsymbol{A}} &= \boldsymbol{A} - (\boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{n})\boldsymbol{n}\otimes\boldsymbol{n} - \tfrac{1}{2}(\mathrm{tr}\,\boldsymbol{A} - \boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{n})(\boldsymbol{I} - \boldsymbol{n}\otimes\boldsymbol{n}), \end{aligned}\right\} \tag{1.7}$$

and, moreover,

$$\left.\begin{aligned} \boldsymbol{u}\wedge\boldsymbol{v} &= \boldsymbol{u}\otimes\boldsymbol{v} - \boldsymbol{v}\otimes\boldsymbol{u}, \\ \boldsymbol{u}\vee\boldsymbol{v} &= \boldsymbol{u}\otimes\boldsymbol{v} + \boldsymbol{v}\otimes\boldsymbol{u}, \end{aligned}\right\} \tag{1.8}$$

$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{r}] = \boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{r}) = \boldsymbol{r}\cdot(\boldsymbol{u}\times\boldsymbol{v}) = \boldsymbol{v}\cdot(\boldsymbol{r}\times\boldsymbol{u}). \tag{1.9}$$

Here and henceforth, the symbols $\boldsymbol{u}\otimes\boldsymbol{v}$ and $\boldsymbol{u}\times\boldsymbol{v}$ are used to denote the tensor product and the vector product of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

2. A unified scheme of constructing functional bases and generating sets

Usually, quite different methods are used to derive functional bases and generating sets separately, which are generally cumbersome. Based upon some recent results by this author (see the related references mentioned before), in this section we shall describe a simple, unified scheme for constructing functional bases and generating sets. Such a scheme enables us to derive irreducible generating sets for general vector-valued and second-order tensor-valued anisotropic functions only from those for vector-valued and second-order tensor-valued isotropic functions with not more than three variables (two variables for the anisotropic functions considered here) and, especially, at the same time it enables us to obtain functional bases for scalar-valued anisotropic functions directly by using generating sets obtained for vector-valued and second-order tensor-valued functions.

First, we show how to construct irreducible generating sets for general vector-valued and second-order tensor-valued anisotropic functions. According to the results for isotropic extension of anisotropic functions given in Xiao (1995*a*, 1996*a*), we can obtain complete generating sets for general vector-valued and second-order tensor-valued anisotropic functions relative to any given anisotropy group $g \subset \mathrm{Orth}$ directly by applying the well-known results for isotropic functions (see Wang 1970; Smith 1971; Boehler 1977; Pennisi & Trovato 1987). To this end, it suffices to investigate anisotropic functions of not more than three variables (see Xiao 1996*b*). The results obtained in such a direct manner, however, need not be irreducible. To arrive at irreducible representations, further consideration is needed. For the sake of definiteness and simplicity, in what follows we shall discuss only the anisotropic functions of interest in this paper. The generalization is direct.

Let g be any subgroup of $C_{\infty h}$. Henceforth, we denote the domain

$$V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$$

by $\mathcal{D}$. For any given set of vectors and second-order tensors,

$$X = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_a; \boldsymbol{W}_1, \ldots, \boldsymbol{W}_b; \boldsymbol{A}_1, \ldots, \boldsymbol{A}_c) \in \mathcal{D},$$

it has been proved (see Xiao 1999) that there are $X_0 \subset X$, where X_0 is a subset of X with not more than two elements, such that

$$g \cap g(X) = g \cap g(X_0). \tag{2.1}$$

Accordingly, an irreducible generating set for general anisotropic functions of the variables X relative to the group $g \subset C_{\infty h}$ can be formed by union of irreducible generating sets for the same type of anisotropic functions of not more than two variables (see Xiao 1999), and hence the problem of determining the former is reduced to that of determining the latter. We shall construct irreducible generating sets for the aforementioned lists of variables, X_0, by means of the following procedures.

1. Construct irreducible generating sets $G(\boldsymbol{u})$, $G(\boldsymbol{W})$ and $G(\boldsymbol{A})$ for a vector variable $\boldsymbol{u}$, a skewsymmetric tensor variable $\boldsymbol{W}$ and a symmetric tensor variable $\boldsymbol{A}$, respectively.

2. For each list $X_0 = (\boldsymbol{x}, \boldsymbol{y})$ of two variables, where $\boldsymbol{x}$ is a vector variable and $\boldsymbol{y}$ is a vector or a second-order tensor variable, form the union $G(\boldsymbol{x}) \cup G(\boldsymbol{y})$. This union obeys the criterion (1.1) for the cases

$$g(\boldsymbol{z}) \cap g = g(X_0) \cap g, \qquad \boldsymbol{z} \in \{\boldsymbol{x}, \boldsymbol{y}\}.$$

Thus, it suffices to treat the case other than the above cases, which is specified by the conditions

$$g(\boldsymbol{z}) \cap g \neq g(X_0) \cap g, \qquad \boldsymbol{z} = \boldsymbol{x}, \boldsymbol{y}. \tag{2.2}$$

Then, analyse the latter case and judge whether or not the aforementioned union also obeys the criterion (1.1) for the latter case. If no, then add some generators with two variables $\boldsymbol{x}$ and $\boldsymbol{y}$ into this union so that an irreducible generating set for the two variables $X_0 = (\boldsymbol{x}, \boldsymbol{y})$ is formed. If yes, then the aforementioned union is already the desired result.

We would point out that the conditions (2.2) usually result in such an $X_0 = (\boldsymbol{x}, \boldsymbol{y})$ that both the construction of the foregoing generators with two variables $\boldsymbol{x}$ and $\boldsymbol{y}$ and the determination of a functional basis of X_0 (see below) can be considerably simplified (e.g. see Cases 4–6 in § 4 *a*).

Next, we show how to obtain a functional basis directly by using the generating sets for vector-valued and second-order tensor-valued functions derived in fulfilling the aforementioned procedures. Our method is mainly based upon the following fact.

Fact 2.1. Let $X \in \mathcal{D}$ be a set of vectors and second-order tensors with a subset X_0 satisfying (2.1). Moreover, let

$$\begin{aligned} V(X_0) &= \{\boldsymbol{h}_1(X_0), \ldots, \boldsymbol{h}_r(X_0)\}, \\ \mathrm{Skw}(X_0) &= \{\boldsymbol{\Omega}_1(X_0), \ldots, \boldsymbol{\Omega}_s(X_0)\}, \\ \mathrm{Sym}(X_0) &= \{\boldsymbol{\Psi}_1(X_0), \ldots, \boldsymbol{\Psi}_t(X_0)\}, \end{aligned}$$

be generating sets for vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the set of variables, X_0, relative to the group g, respectively. Then any vector $\boldsymbol{r} \in X/X_0$, any skewsymmetric tensor $\boldsymbol{H} \in X/X_0$ and any symmetric tensor $\boldsymbol{C} \in X/X_0$ can be determined by the following three sets of invariants:

$$\{\boldsymbol{h}_1(X_0)\cdot\boldsymbol{r}, \ldots, \boldsymbol{h}_r(X_0)\cdot\boldsymbol{r}\},$$
$$\{\operatorname{tr}\boldsymbol{\Omega}_1(X_0)\boldsymbol{H}, \ldots, \operatorname{tr}\boldsymbol{\Omega}_s(X_0)\boldsymbol{H}\},$$
$$\{\operatorname{tr}\boldsymbol{\Psi}_1(X_0)\boldsymbol{C}, \ldots, \operatorname{tr}\boldsymbol{\Psi}_t(X_0)\boldsymbol{C}\}.$$

The proof is as follows. From (2.1) we deduce

$$g(X_0)\cap g \subset g(\boldsymbol{z}), \qquad \boldsymbol{z}\in X.$$

By virtue of this and the obvious fact,

$$g_1 \subset g_2 \Longrightarrow M(g_2) \subset M(g_1),$$

for any two subgroups $g_1, g_2 \subset \mathrm{Orth}$ and $M \subset \{V, \mathrm{Skw}, \mathrm{Sym}\}$, we infer

$$\boldsymbol{r} \in V(g(\boldsymbol{r})) \subset V(g\cap g(X_0)) = \operatorname{span} V(X_0),$$
$$\boldsymbol{H} \in \mathrm{Skw}(g(\boldsymbol{H})) \subset \mathrm{Skw}(g\cap g(X_0)) = \operatorname{span}\mathrm{Skw}(X_0),$$
$$\boldsymbol{C} \in \mathrm{Sym}(g(\boldsymbol{C})) \subset \mathrm{Sym}(g\cap g(X_0)) = \operatorname{span}\mathrm{Sym}(X_0),$$

for any vector $\boldsymbol{r} \in X/X_0$, any skewsymmetric tensor $\boldsymbol{H} \in X/X_0$ and any symmetric tensor $\boldsymbol{C} \in X/X_0$, where the last equality in each of the above three expressions can be derived from (2.13), (2.16) and (2.15) in Xiao (1996*b*). Here $\operatorname{span}\mathcal{S}$ is used to denote the subspace spanned by a set $\mathcal{S}$ of vectors or tensors. Thus, we deduce that the three sets given before, each of which is formed by the invariants obtained by forming the corresponding inner product between each $\boldsymbol{z} \in \{\boldsymbol{r}, \boldsymbol{H}, \boldsymbol{C}\}$ and each generator for X_0, can completely determine $\boldsymbol{r}$, $\boldsymbol{H}$ and $\boldsymbol{C}$. This indicates that the fact mentioned before is true.

Further, let $I(X_0)$ be a functional basis of the subset $X_0 \subset X$ obeying (2.1) relative to the group g. Then, the general set X of variables can be determined up to an orthogonal tensor pertaining to the group g by the basis $I(X_0)$, as well as all invariants given by the inner products in the aforementioned three sets when $\boldsymbol{r}, \boldsymbol{H}, \boldsymbol{A}$ runs over the set X/X_0. This fact is a direct corollary of the fact proved above and the fact (see (1.5)) that the subset X_0 is determined up to an orthogonal tensor pertaining to the group g by the functional basis $I(X_0)$. Thus, applying the just-stated fact and the fact that a set $I(X)$ of invariants under the group g is a functional basis under the group g iff any $X \in \mathcal{D}$ can be determined up to an orthogonal tensor belonging to g by $I(X)$ (see (1.5)), by means of generating sets for vector-valued and second-order tensor-valued functions derived in fulfilling the aforementioned procedures we can obtain functional bases for scalar-valued functions directly from the inner product between each vector generator and a generic vector, the inner product between each tensor generator and a generic tensor, as well as functional bases for X_0. The main procedures are as follows.

1. In accordance with Step 1 stated above, construct invariants by forming the inner product between generic variable $\boldsymbol{z} \in X$ and each given generator, and moreover, construct an irreducible functional basis for one variable.

2. In accordance with Step 2 stated above, construct invariants by forming the inner product between either a generic vector or tensor variable $\boldsymbol{z} \in X$ and each given generator, and moreover, construct an irreducible functional basis for X_0 specified by (2.2).

After fulfilling the above procedure for all X_0 obeying (2.1), the results obtained together provide a complete functional basis. However, the irreducibility of such a basis remains to be proved. In the succeeding sections, we shall construct irreducible representations for scalar-, vector-, and second-order tensor-valued anisotropic functions of any finite number of vector and second-order tensor variables relative to all kinds of subgroups of the transverse isotropy group $C_{\infty h}$ by applying the above unified scheme.

3. The triclinic and monoclinic crystal classes C_1, S_2 and C_{1h}

(*a*) *The triclinic group* C_1

Henceforth, $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is an orthonormal basis of V. Trivially,

V	$\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3,$
Skw	$\boldsymbol{e}_1 \wedge \boldsymbol{e}_2, \boldsymbol{e}_2 \wedge \boldsymbol{e}_3, \boldsymbol{e}_3 \wedge \boldsymbol{e}_1,$
Sym	$\boldsymbol{e}_1 \otimes \boldsymbol{e}_1, \boldsymbol{e}_2 \otimes \boldsymbol{e}_2, \boldsymbol{e}_3 \otimes \boldsymbol{e}_3, \boldsymbol{e}_1 \vee \boldsymbol{e}_2, \boldsymbol{e}_2 \vee \boldsymbol{e}_3, \boldsymbol{e}_3 \vee \boldsymbol{e}_1,$
R	$\boldsymbol{r} \cdot \boldsymbol{e}_i; \boldsymbol{e}_i \cdot \boldsymbol{H}\boldsymbol{e}_j; \boldsymbol{e}_i \cdot \boldsymbol{C}\boldsymbol{e}_j; i, j = 1, 2, 3,$

where $\boldsymbol{r} = \boldsymbol{u}_1, \ldots, \boldsymbol{u}_a, ; \boldsymbol{H} = \boldsymbol{W}_1, \ldots, \boldsymbol{W}_b$ and $\boldsymbol{C} = \boldsymbol{A}_1, \ldots, \boldsymbol{A}_c$, provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables $X \in \mathcal{D}$ relative to the triclinic group C_1.

(*b*) *The triclinic group* S_2

It is evident that for any $X \in \mathcal{D}$ there is a vector $\boldsymbol{u} \in \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_a\} \subset X$ such that

$$g(X) \cap S_2 = g(\boldsymbol{u}) \cap S_2.$$

Accordingly, we construct the following table.

V	$\boldsymbol{u}, \boldsymbol{u} \times \boldsymbol{e}_1, \boldsymbol{u} \times \boldsymbol{e}_2, \boldsymbol{u} \times \boldsymbol{e}_3,$
Skw	$\boldsymbol{e}_1 \wedge \boldsymbol{e}_2, \boldsymbol{e}_2 \wedge \boldsymbol{e}_3, \boldsymbol{e}_3 \wedge \boldsymbol{e}_1,$
Sym	$\boldsymbol{e}_1 \otimes \boldsymbol{e}_1, \boldsymbol{e}_2 \otimes \boldsymbol{e}_2, \boldsymbol{e}_3 \otimes \boldsymbol{e}_3, \boldsymbol{e}_1 \vee \boldsymbol{e}_2, \boldsymbol{e}_2 \vee \boldsymbol{e}_3, \boldsymbol{e}_3 \vee \boldsymbol{e}_1,$
R	$\boldsymbol{u} \cdot \boldsymbol{v}, [\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{e}_i]; \boldsymbol{e}_i \cdot \boldsymbol{H}\boldsymbol{e}_j; \boldsymbol{e}_i \cdot \boldsymbol{C}\boldsymbol{e}_j; \langle(\boldsymbol{u} \cdot \boldsymbol{e}_i)(\boldsymbol{u} \cdot \boldsymbol{e}_j)\rangle; \quad i, j = 1, 2, 3.$

Here and hereafter, the invariants placed in the angle brackets supply an irreducible functional basis for one or two variables under consideration. The other invariants are obtained by forming the corresponding inner product between each generator given and a generic variable $\boldsymbol{x} \in X$. Thus, we obtain the main result of this subsection as follows.

Theorem 3.1. *The table given above, together with*

$$\boldsymbol{u}, \boldsymbol{v} = \boldsymbol{u}_1, \ldots, \boldsymbol{u}_a, \quad \boldsymbol{u} \neq \boldsymbol{v}; \quad \boldsymbol{W} = \boldsymbol{W}_1, \ldots, \boldsymbol{W}_b; \quad \boldsymbol{A} = \boldsymbol{A}_1, \ldots, \boldsymbol{A}_c,$$

provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables $X \in \mathcal{D}$ *under the triclinic group* S_2.

Remarks. An irreducible functional basis of two vector variables under S_2 was derived by Liu (1982), in which 18 invariants are used. Here only 16 invariants are used.

(c) *The monoclinic crystal class* C_{1h}

Since C_{1h} has only two subgroups, i.e. C_1 and itself, we infer that for any $X \in \mathcal{D}$ there exists a single vector or second-order tensor $\boldsymbol{x} \in X$ such that

$$g(X) \cap C_{1h}(\boldsymbol{n}) = g(\boldsymbol{x}) \cap C_{1h}(\boldsymbol{n}).$$

Consequently, it suffices to consider representations concerning one variable. Henceforth, $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are two orthonormal vectors in the $\boldsymbol{n}$-plane and

$$\boldsymbol{N} = \boldsymbol{e}_1 \wedge \boldsymbol{e}_2, \tag{3.1}$$

where $\boldsymbol{N}$ is invariant under $C_{\infty h}(\boldsymbol{n})$ and independent of choice of the orthonormal vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ in the $\boldsymbol{n}$-plane.

Case 1. $\exists \boldsymbol{u} \in X : g(X) \cap C_{1h}(\boldsymbol{n}) = g(\boldsymbol{u}) \cap C_{1h}(\boldsymbol{n})$.

$$\begin{array}{rl}
V & \boldsymbol{e}_1, \boldsymbol{e}_2, (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n},\\
\mathrm{Skw} & \boldsymbol{N}, (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}\wedge\boldsymbol{e}_1, (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}\wedge\boldsymbol{e}_2,\\
\mathrm{Sym} & \boldsymbol{n}\otimes\boldsymbol{n}, \boldsymbol{e}_1\otimes\boldsymbol{e}_1, \boldsymbol{e}_2\otimes\boldsymbol{e}_2, \boldsymbol{e}_1\vee\boldsymbol{e}_2, (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}\vee\boldsymbol{e}_1, (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}\vee\boldsymbol{e}_2,\\
R & \boldsymbol{r}\cdot\boldsymbol{e}_1, \boldsymbol{r}\cdot\boldsymbol{e}_2, (\boldsymbol{r}\cdot\boldsymbol{n})(\boldsymbol{u}\cdot\boldsymbol{n}),\\
 & \mathrm{tr}\,\boldsymbol{HN}, (\boldsymbol{u}\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{He}_1), (\boldsymbol{u}\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{He}_2),\\
 & \boldsymbol{n}\cdot\boldsymbol{Cn}, \boldsymbol{e}_1\cdot\boldsymbol{Ce}_1, \boldsymbol{e}_2\cdot\boldsymbol{Ce}_2, \boldsymbol{e}_1\cdot\boldsymbol{Ce}_2, (\boldsymbol{u}\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{Ce}_1), (\boldsymbol{u}\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{Ce}_2),\\
 & \langle \boldsymbol{u}\cdot\boldsymbol{e}_1, \boldsymbol{u}\cdot\boldsymbol{e}_2, (\boldsymbol{u}\cdot\boldsymbol{n})^2\rangle.
\end{array}$$

Case 2. $\exists \boldsymbol{W} \in X : g(X) \cap C_{1h}(\boldsymbol{n}) = g(\boldsymbol{W}) \cap C_{1h}(\boldsymbol{n})$.

$$\begin{array}{rl}
V & \boldsymbol{e}_1, \boldsymbol{e}_2, (\boldsymbol{n}\cdot\boldsymbol{We}_1)\boldsymbol{n}, (\boldsymbol{n}\cdot\boldsymbol{We}_2)\boldsymbol{n},\\
\mathrm{Skw} & \boldsymbol{N}, \boldsymbol{n}\wedge\boldsymbol{Wn}, \boldsymbol{n}\wedge(\boldsymbol{n}\times\boldsymbol{Wn}),\\
\mathrm{Sym} & \boldsymbol{n}\otimes\boldsymbol{n}, \boldsymbol{e}_1\otimes\boldsymbol{e}_1, \boldsymbol{e}_2\otimes\boldsymbol{e}_2, \boldsymbol{e}_1\vee\boldsymbol{e}_2, \boldsymbol{n}\vee\boldsymbol{Wn}, \boldsymbol{n}\vee(\boldsymbol{n}\times\boldsymbol{Wn}),\\
R & \boldsymbol{r}\cdot\boldsymbol{e}_1, \boldsymbol{r}\cdot\boldsymbol{e}_2, (\boldsymbol{r}\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{We}_1), (\boldsymbol{r}\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{We}_2),\\
 & \mathrm{tr}\,\boldsymbol{HN}, \boldsymbol{n}\cdot\boldsymbol{WHn}, [\boldsymbol{n}, \boldsymbol{Wn}, \boldsymbol{Hn}],\\
 & \boldsymbol{n}\cdot\boldsymbol{Cn}, \boldsymbol{e}_1\cdot\boldsymbol{Ce}_1, \boldsymbol{e}_2\cdot\boldsymbol{Ce}_2, \boldsymbol{e}_1\cdot\boldsymbol{Ce}_2, \boldsymbol{n}\cdot\boldsymbol{WCn}, [\boldsymbol{n}, \boldsymbol{Wn}, \boldsymbol{Cn}],\\
 & \langle \boldsymbol{e}_1\cdot\boldsymbol{We}_2, (\boldsymbol{n}\cdot\boldsymbol{We}_1)^2, (\boldsymbol{n}\cdot\boldsymbol{We}_2)^2, (\boldsymbol{n}\cdot\boldsymbol{We}_1)(\boldsymbol{n}\cdot\boldsymbol{We}_2)\rangle.
\end{array}$$

It can readily be verified that the first three sets given above are irreducible generating sets for vector-valued, skewsymmetric and symmetric tensor-valued anisotropic functions of the skewsymmetric tensor variable $\boldsymbol{W} \in \mathrm{Skw}$ relative to $C_{1h}(\boldsymbol{n})$, respectively, by means of (1.2)–(1.4) and the fact

$$g(\boldsymbol{W}) \cap C_{1h}(\boldsymbol{n}) = \begin{cases} C_{1h}(\boldsymbol{n}), & \boldsymbol{Wn} = \boldsymbol{0},\\ C_1, & \boldsymbol{Wn} \neq \boldsymbol{0}. \end{cases}$$

Moreover, it can be proved easily that the four invariants of $\boldsymbol{W}$ listed in the angle brackets form an irreducible functional basis of the variable $\boldsymbol{W} \in \mathrm{Skw}$ under $C_{1h}(\boldsymbol{n})$.

Case 3. $\exists \boldsymbol{A} \in X : g(X) \cap C_{1h}(\boldsymbol{n}) = g(\boldsymbol{A}) \cap C_{1h}$.

$$\begin{array}{rl}
V & \boldsymbol{e}_1, \boldsymbol{e}_2, (\boldsymbol{n}\cdot\boldsymbol{Ae}_1)\boldsymbol{n}, (\boldsymbol{n}\cdot\boldsymbol{Ae}_2)\boldsymbol{n},\\
\mathrm{Skw} & \boldsymbol{N}, \boldsymbol{n}\wedge\boldsymbol{An}, \boldsymbol{n}\wedge(\boldsymbol{n}\times\boldsymbol{An}),
\end{array}$$

$$\begin{array}{ll} \mathrm{Sym} & \boldsymbol{n}\otimes\boldsymbol{n}, \boldsymbol{e}_1\otimes\boldsymbol{e}_1, \boldsymbol{e}_2\otimes\boldsymbol{e}_2, \boldsymbol{e}_1\vee\boldsymbol{e}_2, \boldsymbol{n}\vee\boldsymbol{A}\boldsymbol{n}, \boldsymbol{n}\vee(\boldsymbol{n}\times\boldsymbol{A}\boldsymbol{n}), \\ R & \boldsymbol{r}\cdot\boldsymbol{e}_1, \boldsymbol{r}\cdot\boldsymbol{e}_2, (\boldsymbol{r}\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{e}_1), (\boldsymbol{r}\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{e}_2), \\ & \mathrm{tr}\,\boldsymbol{H}\boldsymbol{N}, \boldsymbol{n}\cdot\boldsymbol{H}\boldsymbol{A}\boldsymbol{n}, [\boldsymbol{n}, \boldsymbol{A}\boldsymbol{n}, \boldsymbol{H}\boldsymbol{n}], \\ & \boldsymbol{n}\cdot\boldsymbol{C}\boldsymbol{n}, \boldsymbol{e}_1\cdot\boldsymbol{C}\boldsymbol{e}_1, \boldsymbol{e}_2\cdot\boldsymbol{C}\boldsymbol{e}_2, \boldsymbol{e}_1\cdot\boldsymbol{C}\boldsymbol{e}_2, \boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{C}\boldsymbol{n}, [\boldsymbol{n}, \boldsymbol{A}\boldsymbol{n}, \boldsymbol{C}\boldsymbol{n}], \\ & \langle \boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{n}, \boldsymbol{e}_1\cdot\boldsymbol{A}\boldsymbol{e}_1, \boldsymbol{e}_2\cdot\boldsymbol{A}\boldsymbol{e}_2, \boldsymbol{e}_1\cdot\boldsymbol{A}\boldsymbol{e}_2, \\ & \quad (\boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{e}_1)^2, (\boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{e}_2)^2, (\boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{e}_1)(\boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{e}_2)\rangle. \end{array}$$

By means of the criterion (1.1) and the equalities (1.2)–(1.4) and the fact

$$g(\boldsymbol{A})\cap C_{1h}(\boldsymbol{n}) = \begin{cases} C_{1h}(\boldsymbol{n}), & \boldsymbol{n}\times\boldsymbol{A}\boldsymbol{n}=\boldsymbol{0}, \\ C_1, & \boldsymbol{n}\times\boldsymbol{A}\boldsymbol{n}\neq\boldsymbol{0}, \end{cases}$$

it can readily be checked that the first three sets given above are irreducible generating sets for vector-valued, skewsymmetric and symmetric tensor-valued anisotropic functions of the symmetric tensor variable $\boldsymbol{A}\in\mathrm{Sym}$ relative to $C_{1h}(\boldsymbol{n})$, respectively. Moreover, the irreducible functional basis of the symmetric tensor variable $\boldsymbol{A}\in\mathrm{Sym}$ under $C_{1h}(\boldsymbol{n})$, listed in the angle brackets, is cited from Xiao (1996*c*).

Combining the above cases, we arrive at the main result of this subsection as follows.

Theorem 3.2. *The four sets given by*

$$\boldsymbol{u}\cdot\boldsymbol{e}_1, \boldsymbol{u}\cdot\boldsymbol{e}_2, (\boldsymbol{u}\cdot\boldsymbol{n})^2; \boldsymbol{e}_1\cdot\boldsymbol{W}\boldsymbol{e}_2, (\boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{e}_1)^2, (\boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{e}_2)^2, (\boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{e}_1)(\boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{e}_2);$$
$$\boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{n}, \boldsymbol{e}_1\cdot\boldsymbol{A}\boldsymbol{e}_1, \boldsymbol{e}_2\cdot\boldsymbol{A}\boldsymbol{e}_2, \boldsymbol{e}_1\cdot\boldsymbol{A}\boldsymbol{e}_2, (\boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{e}_1)^2, (\boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{e}_2)^2, (\boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{e}_1)(\boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{e}_2);$$
$$(\boldsymbol{u}\cdot\boldsymbol{n})(\boldsymbol{v}\cdot\boldsymbol{n}); (\boldsymbol{u}\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{e}_1), (\boldsymbol{u}\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{e}_2);$$
$$\boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{H}\boldsymbol{n}, [\boldsymbol{n}, \boldsymbol{W}\boldsymbol{n}, \boldsymbol{H}\boldsymbol{n}]; \boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{B}\boldsymbol{n}, [\boldsymbol{n}, \boldsymbol{A}\boldsymbol{n}, \boldsymbol{B}\boldsymbol{n}]; \boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{W}\boldsymbol{n}, [\boldsymbol{n}, \boldsymbol{W}\boldsymbol{n}, \boldsymbol{A}\boldsymbol{n}];$$

and

$$\boldsymbol{e}_1, \boldsymbol{e}_2, (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}; (\boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{e}_1)\boldsymbol{n}, (\boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{e}_2)\boldsymbol{n}; (\boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{e}_1)\boldsymbol{n}, (\boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{e}_2)\boldsymbol{n};$$

and

$$\boldsymbol{e}_1\wedge\boldsymbol{e}_2, (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}\wedge\boldsymbol{e}_1, (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}\wedge\boldsymbol{e}_2; \boldsymbol{n}\wedge\boldsymbol{W}\boldsymbol{n}, \boldsymbol{n}\wedge(\boldsymbol{n}\times\boldsymbol{W}\boldsymbol{n}); \boldsymbol{n}\wedge\boldsymbol{A}\boldsymbol{n}, \boldsymbol{n}\wedge(\boldsymbol{n}\times\boldsymbol{A}\boldsymbol{n});$$

and

$$\boldsymbol{e}_1\otimes\boldsymbol{e}_1, \boldsymbol{e}_2\otimes\boldsymbol{e}_2, \boldsymbol{n}\otimes\boldsymbol{n}, \boldsymbol{e}_1\vee\boldsymbol{e}_2, (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}\vee\boldsymbol{e}_1, (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}\vee\boldsymbol{e}_2;$$
$$\boldsymbol{n}\vee\boldsymbol{W}\boldsymbol{n}, \boldsymbol{n}\vee(\boldsymbol{n}\times\boldsymbol{W}\boldsymbol{n}); \boldsymbol{n}\vee\boldsymbol{A}\boldsymbol{n}, \boldsymbol{n}\vee(\boldsymbol{n}\times\boldsymbol{A}\boldsymbol{n});$$

where $\boldsymbol{u},\boldsymbol{v}=\boldsymbol{u}_1,\ldots,\boldsymbol{u}_a$; $\boldsymbol{W},\boldsymbol{H}=\boldsymbol{W}_1,\ldots,\boldsymbol{W}_b$; $\boldsymbol{A},\boldsymbol{B}=\boldsymbol{A}_1,\ldots,\boldsymbol{A}_c$; $\boldsymbol{u}\neq\boldsymbol{v}$, $\boldsymbol{W}\neq\boldsymbol{H}$, $\boldsymbol{A}\neq\boldsymbol{B}$, *provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables* $X\in\mathcal{D}$ *under the monoclinic group* $C_{1h}(\boldsymbol{n})$, *respectively.*

4. The classes C_{2mh} and C_{2m}

The classes C_{2mh} and C_{2m} include the monoclinic crystal classes C_{2h} and C_2, the tetrahedral crystal classes C_{4h} and C_4, the hexagonal crystal classes C_{6h} and C_6, as well as the transverse isotropy groups $C_{\infty h}$ and C_∞, as the particular cases when $m=1,2,3,\infty$.

(a) The classes C_{2mh}

According to lemma 2.5 in Xiao (1999), for any $X \in \mathcal{D}$ there exists $X_0 \subset X$ such that

$$g(X) \cap C_{2mh}(\boldsymbol{n}) = g(X_0) \cap C_{2mh}(\boldsymbol{n}),$$

where $X_0 \in \{(\boldsymbol{u}, \boldsymbol{v}), (\boldsymbol{u}, \boldsymbol{W}), (\boldsymbol{u}, \boldsymbol{A})\}$. Thus, following the scheme outlined in § 2, six cases are discussed as follows.

Case 1. $\exists \boldsymbol{u} \in X : g(X) \cap C_{2mh}(\boldsymbol{n}) = g(\boldsymbol{u}) \cap C_{2mh}(\boldsymbol{n})$.

From the criterion (1.1) and the formulae (1.2)–(1.4), as well as

$$g(\boldsymbol{u}) \cap C_{2mh}(\boldsymbol{n}) = \begin{cases} C_{2mh}(\boldsymbol{n}), & \boldsymbol{u} = \boldsymbol{0}, \\ C_{2m}(\boldsymbol{n}), & \boldsymbol{u} = a\boldsymbol{n} \neq \boldsymbol{0}, \\ C_{1h}(\boldsymbol{n}), & \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \boldsymbol{u} \neq \boldsymbol{0}, \\ C_1, & (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{u} \times \boldsymbol{n} \neq \boldsymbol{0}, \end{cases} \tag{4.1}$$

$$g(\boldsymbol{u}) \cap C_{\infty h}(\boldsymbol{n}) = \begin{cases} C_{\infty h}(\boldsymbol{n}), & \boldsymbol{u} = \boldsymbol{0}, \\ C_{\infty}(\boldsymbol{n}), & \boldsymbol{u} = a\boldsymbol{n} \neq \boldsymbol{0}, \\ C_{1h}(\boldsymbol{n}), & \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \boldsymbol{u} \neq \boldsymbol{0}, \\ C_1, & (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{u} \times \boldsymbol{n} \neq \boldsymbol{0}, \end{cases} \tag{4.2}$$

we deduce that the following fact holds: generating sets for vector-valued (for $m \geqslant 1$), skewsymmetric tensor-valued (for $m \geqslant 1$) and symmetric tensor-valued (for $m \geqslant 2$) anisotropic functions of the vector variable $\boldsymbol{u}$ under $C_{2mh}(\boldsymbol{n})$ can be derived from those for vector-, skewsymmetric and symmetric tensor-valued isotropic functions of the extended variables $(\boldsymbol{u}, \boldsymbol{N})$, respectively. Moreover, with the aid of the fact that $(-\boldsymbol{I})\boldsymbol{u} = -\boldsymbol{u}$, $-\boldsymbol{I} \in C_{2mh}(\boldsymbol{n})$ and the invariance condition stated at the start of the introduction, it may readily be understood that functional bases for scalar-valued anisotropic functions of the vector variable $\boldsymbol{u}$ under $C_{2mh}(\boldsymbol{n})$ can be obtained from those for scalar-valued anisotropic functions of the symmetric tensor variable $\boldsymbol{u} \otimes \boldsymbol{u} \in \mathrm{Sym}$ under $C_{2mh}(\boldsymbol{n})$. The latter can be found in Xiao (1996*d*) for $m = 1$ and in Case 3 below for $m \geqslant 2$.

Taking the above facts into account, we construct the following table:

V	$\mathring{\boldsymbol{u}}, \mathring{\boldsymbol{u}} \times \boldsymbol{n}, (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n},$
Skw	$\boldsymbol{N}, (\boldsymbol{u} \cdot \boldsymbol{n})\mathring{\boldsymbol{u}} \wedge \boldsymbol{n}, (\boldsymbol{u} \cdot \boldsymbol{n})\mathring{\boldsymbol{u}} \wedge (\mathring{\boldsymbol{u}} \times \boldsymbol{n}),$
Sym	$\boldsymbol{n} \otimes \boldsymbol{n}; \delta_{1m}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1, \delta_{1m}\boldsymbol{e}_2 \otimes \boldsymbol{e}_2, \delta_{1m}\boldsymbol{e}_1 \otimes \boldsymbol{e}_2; (1-\delta_{1m})\boldsymbol{I},$
	$(1-\delta_{1m})\mathring{\boldsymbol{u}} \otimes \mathring{\boldsymbol{u}}, (1-\delta_{1m})\mathring{\boldsymbol{u}} \vee (\mathring{\boldsymbol{u}} \times \boldsymbol{n}); (\boldsymbol{u} \cdot \boldsymbol{n})\mathring{\boldsymbol{u}} \vee \boldsymbol{n}, (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n} \vee (\mathring{\boldsymbol{u}} \times \boldsymbol{n}),$
R	$\mathring{\boldsymbol{u}} \cdot \mathring{\boldsymbol{r}}, [\boldsymbol{n}, \mathring{\boldsymbol{u}}, \mathring{\boldsymbol{r}}], (\boldsymbol{u} \cdot \boldsymbol{n})(\boldsymbol{r} \cdot \boldsymbol{n}),$
	$\mathrm{tr}\,\boldsymbol{H}\boldsymbol{N}, (\boldsymbol{u} \cdot \boldsymbol{n})\mathring{\boldsymbol{u}} \cdot \boldsymbol{H}\boldsymbol{n}, (\boldsymbol{u} \cdot \boldsymbol{n})[\boldsymbol{n}, \mathring{\boldsymbol{u}}, \boldsymbol{H}\boldsymbol{n}],$
	$\boldsymbol{n} \cdot \boldsymbol{C}\boldsymbol{n}; \delta_{1m}\boldsymbol{e}_1 \cdot \boldsymbol{C}\boldsymbol{e}_1, \delta_{1m}\boldsymbol{e}_2 \cdot \boldsymbol{C}\boldsymbol{e}_2, \delta_{1m}\boldsymbol{e}_1 \cdot \boldsymbol{C}\boldsymbol{e}_2; (1-\delta_{1m})\,\mathrm{tr}\,\boldsymbol{C},$
	$(1-\delta_{1m})\mathring{\boldsymbol{u}} \cdot \boldsymbol{C}\mathring{\boldsymbol{u}}, (1-\delta_{1m})[\boldsymbol{n}, \mathring{\boldsymbol{u}}, \boldsymbol{C}\mathring{\boldsymbol{u}}]; (\boldsymbol{u} \cdot \boldsymbol{n})\mathring{\boldsymbol{u}} \cdot \boldsymbol{C}\boldsymbol{n}, (\boldsymbol{u} \cdot \boldsymbol{n})[\boldsymbol{n}, \boldsymbol{C}\boldsymbol{n}, \mathring{\boldsymbol{u}}],$
	$\langle (\boldsymbol{u} \cdot \boldsymbol{n})^2; \delta_{1m}(\boldsymbol{u} \cdot \boldsymbol{e}_1)^2, \delta_{1m}(\boldsymbol{u} \cdot \boldsymbol{e}_2)^2, \delta_{1m}(\boldsymbol{u} \cdot \boldsymbol{e}_1)(\boldsymbol{u} \cdot \boldsymbol{e}_2);$
	$(1-\delta_{1m})\alpha_m(\mathring{\boldsymbol{u}}), (1-\delta_{1m})\alpha'_m(\mathring{\boldsymbol{u}})\rangle.$

Here and henceforth,

$$\left.\begin{aligned} \alpha_m(\mathring{\boldsymbol{u}}) &= \sum_{r=1}^{m}(\mathring{\boldsymbol{u}}\cdot\boldsymbol{l}_r)^{2m}, \quad \boldsymbol{l}_r = \boldsymbol{R}_{\boldsymbol{n}}^{4r\pi/4m}\boldsymbol{e},\\ \alpha'_m(\mathring{\boldsymbol{u}}) &= \sum_{r=1}^{m}(\mathring{\boldsymbol{u}}\cdot\boldsymbol{l}'_r)^{2m}, \quad \boldsymbol{l}'_r = \boldsymbol{R}_{\boldsymbol{n}}^{(4r+1)\pi/4m}\boldsymbol{e}, \end{aligned}\right\} \tag{4.3}$$

for $m = 2, 3, \ldots$. Here and henceforth, $\boldsymbol{e}$ is used to denote a given unit vector in the $\boldsymbol{n}$-plane.

Henceforth, the first three sets consisting of vector generators, skewsymmetric tensor generators and symmetric tensor generators are represented by $V^0(\boldsymbol{u})$, $\mathrm{Skw}^0(\boldsymbol{u})$ and $\mathrm{Sym}^0_m(\boldsymbol{u})$, respectively, and the set of invariants given in the angle brackets is signified by $I^0_m(\boldsymbol{u})$.

It should be pointed out that in the above table each element with the coefficient δ_{1m} or $(1-\delta_{1m})$ comes into play only when $m = 1$ or $m \geqslant 2$. Throughout, δ_{rs} is used to denote the Kronecker delta. Such a difference between C_{2h} and C_{2mh} for $m \geqslant 2$, which will also appear in the next four cases, arises from the fact

$$\dim \mathrm{Sym}(C_{2mh}) = \begin{cases} \dim \mathrm{Sym}(C_{2h}) = 4, & m = 1,\\ \dim \mathrm{Sym}(C_{2mh}) = 2, & m \geqslant 2. \end{cases}$$

Case 2. $\exists \boldsymbol{W} \in X : g(X) \cap C_{2mh}(\boldsymbol{n}) = g(\boldsymbol{W}) \cap C_{2mh}(\boldsymbol{n})$.

Every vector-valued function of the variable $\boldsymbol{W} \in \mathrm{Skw}$ that is form-invariant under C_{2mh} vanishes.

From the criterion (1.1) and the equalities (1.3)–(1.4), as well as

$$g(\boldsymbol{W}) \cap C_{2mh}(\boldsymbol{n}) = \begin{cases} C_{2mh}(\boldsymbol{n}), & \boldsymbol{W}\boldsymbol{n} = \boldsymbol{0},\\ S_2, & \boldsymbol{W}\boldsymbol{n} \neq \boldsymbol{0}, \end{cases} \tag{4.4}$$

$$g(\boldsymbol{W}) \cap g(\boldsymbol{N}) = \begin{cases} C_{\infty h}(\boldsymbol{n}), & \boldsymbol{W}\boldsymbol{n} = \boldsymbol{0},\\ S_2, & \boldsymbol{W}\boldsymbol{n} \neq \boldsymbol{0}, \end{cases} \tag{4.5}$$

we infer that the following fact holds: generating sets for skewsymmetric tensor-valued (for $m \geqslant 1$) and symmetric tensor-valued (for $m \geqslant 2$) anisotropic functions of the variable $\boldsymbol{W} \in \mathrm{Skw}$ under $C_{2mh}(\boldsymbol{n})$ can be derived from those for skewsymmetric and symmetric tensor-valued isotropic functions of the extended variables $(\boldsymbol{W}, \boldsymbol{N})$. Thus, we construct the following table:

Skw	$\boldsymbol{N}, \boldsymbol{n}\wedge\boldsymbol{W}\boldsymbol{n}, \boldsymbol{n}\wedge(\boldsymbol{n}\times\boldsymbol{W}\boldsymbol{n}),$
Sym	$\boldsymbol{n}\otimes\boldsymbol{n}; \delta_{1m}\boldsymbol{e}_1\otimes\boldsymbol{e}_1, \delta_{1m}\boldsymbol{e}_2\otimes\boldsymbol{e}_2, \delta_{1m}\boldsymbol{e}_1\vee\boldsymbol{e}_2; (1-\delta_{1m})\boldsymbol{I},$
	$(1-\delta_{1m})\boldsymbol{W}\boldsymbol{n}\otimes\boldsymbol{W}\boldsymbol{n}, (1-\delta_{1m})\boldsymbol{W}\boldsymbol{n}\otimes(\boldsymbol{n}\times\boldsymbol{W}\boldsymbol{n});$
	$\boldsymbol{n}\vee\boldsymbol{W}\boldsymbol{n}, \boldsymbol{n}\vee(\boldsymbol{n}\times\boldsymbol{W}\boldsymbol{n}),$
R	$\mathrm{tr}\,\boldsymbol{H}\boldsymbol{N}, \boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{H}\boldsymbol{n}, [\boldsymbol{n}, \boldsymbol{W}\boldsymbol{n}, \boldsymbol{H}\boldsymbol{n}],$
	$\boldsymbol{n}\cdot\boldsymbol{C}\boldsymbol{n}, \delta_{1m}\boldsymbol{e}_1\cdot\boldsymbol{C}\boldsymbol{e}_1, \delta_{1m}\boldsymbol{e}_2\cdot\boldsymbol{C}\boldsymbol{e}_2, \delta_{1m}\boldsymbol{e}_1\cdot\boldsymbol{C}\boldsymbol{e}_2;$
	$(1-\delta_{1m})\boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{C}\boldsymbol{W}\boldsymbol{n}, (1-\delta_{1m})[\boldsymbol{n}, \boldsymbol{W}\boldsymbol{n}, \boldsymbol{W}\boldsymbol{C}\boldsymbol{n}]; \boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{C}\boldsymbol{n}, [\boldsymbol{n}, \boldsymbol{W}\boldsymbol{n}, \boldsymbol{C}\boldsymbol{n}],$
	$\langle \mathrm{tr}\,\boldsymbol{W}\boldsymbol{N}; \delta_{1m}(\boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{e}_1)^2, \delta_{1m}(\boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{e}_2)^2, \delta_{1m}(\boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{e}_1)(\boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{e}_2);$
	$(1-\delta_{1m})\alpha_m(\boldsymbol{W}\boldsymbol{n}), (1-\delta_{1m})\alpha'_m(\boldsymbol{W}\boldsymbol{n})\rangle.$

Henceforth, the two generating sets in the above table are denoted by $\mathrm{Skw}^0(\boldsymbol{W})$ and $\mathrm{Sym}^0_m(\boldsymbol{W})$, and the functional basis given in the angle brackets is designated by $I^0_m(\boldsymbol{W})$.

We need only to show that the set $I^0_m(\boldsymbol{W})$ is an irreducible functional basis of $\boldsymbol{W}$ under $C_{2mh}(\boldsymbol{n})$. To this end, we prove that $I^0_m(\boldsymbol{W})$ obeys the criterion (1.5). The proof for $m = 1$ is easy. Let $m \geqslant 2$. Observing the fact that the last two invariants in the above table form a functional basis of the vector $\boldsymbol{Wn}$ in the $\boldsymbol{n}$-plane (see Case 1), we infer that for $\bar{\boldsymbol{W}}, \boldsymbol{W} \in \mathrm{Skw}$,

$$I^0_m(\bar{\boldsymbol{W}}) = I^0_m(\boldsymbol{W}) \Longrightarrow \exists \boldsymbol{Q} \in C_{2mh}(\boldsymbol{n}) : \bar{\boldsymbol{W}}\boldsymbol{n} = \boldsymbol{Q}(\boldsymbol{Wn}), \mathrm{tr}\, \bar{\boldsymbol{W}}\boldsymbol{N} = \mathrm{tr}\, \boldsymbol{WN}.$$

In the above, we can assume $\boldsymbol{Q} \in C_{2m}(\boldsymbol{n})$, since $\boldsymbol{Wn}$ lies in the $\boldsymbol{n}$-plane. Thus, by means of the identity

$$\boldsymbol{W} = \tfrac{1}{2}(\mathrm{tr}\, \boldsymbol{WN})\boldsymbol{N} + (\boldsymbol{Wn}) \wedge \boldsymbol{n}, \tag{4.6}$$

as well as the facts: $\boldsymbol{QNQ}^{\mathrm{T}} = \boldsymbol{N}$, $\boldsymbol{Qn} = \boldsymbol{n}$ for each $\boldsymbol{Q} \in C_{2m}(\boldsymbol{n})$, we deduce

$$I^0_m(\bar{\boldsymbol{W}}) = I^0_m(\boldsymbol{W}) \Longrightarrow \exists \boldsymbol{Q} \in C_{2m}(\boldsymbol{n}) : \bar{\boldsymbol{W}} = \boldsymbol{QWQ}^{\mathrm{T}}.$$

Thus, we conclude that $I^0_m(\boldsymbol{W})$ obeys the criterion (1.5). The irreducibility of this basis is evident.

Case 3. $\exists \boldsymbol{A} \in X : g(X) \cap C_{2mh}(\boldsymbol{n}) = g(\boldsymbol{A}) \cap C_{2mh}(\boldsymbol{n})$.

Every vector-valued function of the symmetric tensor variable $\boldsymbol{A} \in \mathrm{Sym}$ that is form-invariant under $C_{2mh}(\boldsymbol{n})$ vanishes.

From the facts

$$g(\boldsymbol{A}) \cap C_{2mh}(\boldsymbol{n}) = \begin{cases} C_{2mh}(\boldsymbol{n}), & \mathring{\boldsymbol{A}} = \boldsymbol{O}, \\ C_{2h}(\boldsymbol{n}), & \boldsymbol{n} \times \boldsymbol{An} = \boldsymbol{0}, \quad \mathring{\boldsymbol{A}} \neq \boldsymbol{O}, \\ S_2, & \boldsymbol{n} \times \boldsymbol{An} \neq \boldsymbol{0}, \end{cases} \tag{4.7}$$

$$g(\boldsymbol{A}) \cap g(\boldsymbol{N}) = \begin{cases} C_{\infty h}(\boldsymbol{n}), & \mathring{\boldsymbol{A}} = \boldsymbol{O}, \\ C_{2h}(\boldsymbol{n}), & \boldsymbol{n} \times \boldsymbol{An} = \boldsymbol{0}, \quad \mathring{\boldsymbol{A}} \neq \boldsymbol{O}, \\ S_2, & \boldsymbol{n} \times \boldsymbol{An} \neq \boldsymbol{0}, \end{cases} \tag{4.8}$$

and the criterion (1.1), as well as the equalities (1.3)–(1.4), we infer that the following fact holds: generating sets for skewsymmetric tensor-valued (for $m \geqslant 1$) and symmetric tensor-valued (for $m \geqslant 2$) anisotropic functions of the variable $\boldsymbol{A} \in \mathrm{Skw}$ under $C_{2mh}(\boldsymbol{n})$ can be derived from those for skewsymmetric and symmetric tensor-valued isotropic functions of the extended variables $(\boldsymbol{W}, \boldsymbol{N})$. Thus, we construct the following table:

Skw	$\boldsymbol{N}, \boldsymbol{n} \wedge \boldsymbol{An}, \boldsymbol{n} \wedge (\boldsymbol{n} \times \boldsymbol{An}),$
Sym	$\boldsymbol{n} \otimes \boldsymbol{n}; \delta_{1m}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1, \delta_{1m}\boldsymbol{e}_2 \otimes \boldsymbol{e}_2, \delta_{1m}\boldsymbol{e}_1 \vee \boldsymbol{e}_2; (1-\delta_{1m})\boldsymbol{I},$
	$(1-\delta_{1m})\mathring{\boldsymbol{A}}\boldsymbol{n} \otimes \mathring{\boldsymbol{A}}\boldsymbol{n}, (1-\delta_{1m})\mathring{\boldsymbol{A}}\boldsymbol{n} \vee (\boldsymbol{n} \times \mathring{\boldsymbol{A}}\boldsymbol{n}); \boldsymbol{A}, \boldsymbol{AN} - \boldsymbol{NA},$
R	$\mathrm{tr}\, \boldsymbol{HN}, \boldsymbol{n} \cdot \boldsymbol{HAn}, [\boldsymbol{n}, \boldsymbol{An}, \boldsymbol{Hn}],$
	$\boldsymbol{n} \cdot \boldsymbol{Cn}, \mathrm{tr}\, \boldsymbol{AC}, \mathrm{tr}\, \boldsymbol{ACN}; \delta_{1m}\boldsymbol{e}_1 \cdot \boldsymbol{Ce}_1, \delta_{1m}\boldsymbol{e}_2 \cdot \boldsymbol{Ce}_2, \delta_{1m}\boldsymbol{e}_1 \cdot \boldsymbol{Ce}_2;$
	$(1-\delta_{1m})\, \mathrm{tr}\, \boldsymbol{C}, (1-\delta_{1m})\boldsymbol{n} \cdot \mathring{\boldsymbol{A}}\boldsymbol{C}\mathring{\boldsymbol{A}}\boldsymbol{n}, (1-\delta_{1m})[\boldsymbol{n}, \mathring{\boldsymbol{A}}\boldsymbol{n}, \boldsymbol{C}\mathring{\boldsymbol{A}}\boldsymbol{n}]$
	$\langle \boldsymbol{n} \cdot \boldsymbol{An}, [\boldsymbol{n}, \boldsymbol{An}, \boldsymbol{A}^2\boldsymbol{n}]; \delta_{1m}\boldsymbol{e}_1 \cdot \boldsymbol{Ae}_1, \delta_{1m}\boldsymbol{e}_2 \cdot \boldsymbol{Ae}_2, \delta_{1m}\boldsymbol{e}_1 \cdot \boldsymbol{Ae}_2,$

$$\delta_{1m}(\boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{e}_1)^2, \delta_{1m}(\boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{e}_2)^2, \delta_{1m}(\boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{e}_1)(\boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{e}_2); (1-\delta_{1m})\,\mathrm{tr}\,\boldsymbol{A},$$
$$(1-\delta_{1m})\,\mathrm{tr}\,\boldsymbol{A}^3, (1-\delta_{1m})\alpha_m(\mathring{\boldsymbol{A}}\boldsymbol{n}), (1-\delta_{1m})\alpha'_m(\mathring{\boldsymbol{A}}\boldsymbol{n}),$$
$$(1-\delta_{1m})\beta_m(\mathring{\boldsymbol{A}}), (1-\delta_{1m})\beta'_m(\mathring{\boldsymbol{A}})\rangle.$$

In the above, the invariants $\alpha'_m(\mathring{\boldsymbol{A}}\boldsymbol{n})$ and $\alpha'(\mathring{\boldsymbol{A}}\boldsymbol{n})$ are obtained by replacing $\mathring{\boldsymbol{u}}$ with $\mathring{\boldsymbol{A}}\boldsymbol{n}$ in (4.3), and moreover,

$$\beta_m(\mathring{\boldsymbol{A}}) = \sum_{r=1}^{m}(\boldsymbol{l}_r\cdot\mathring{\boldsymbol{A}}\boldsymbol{l}_r)^m, \qquad \beta'_m(\mathring{\boldsymbol{A}}) = \sum_{r=1}^{m}(\boldsymbol{l}'_r\cdot\mathring{\boldsymbol{A}}\boldsymbol{l}'_r)^m, \tag{4.9}$$

where the unit vectors $\boldsymbol{l}_r$ and $\boldsymbol{l}'_r$ lie in the $\boldsymbol{n}$-plane and are given in (4.3).

It can easily be shown that the presented generating set for skewsymmetric tensor-valued functions, denoted by $\mathrm{Skw}^0(\boldsymbol{A})$ henceforth, obeys the criterion (1.1). It is evident that this set is irreducible. Moreover, the presented generating set for symmetric tensor-valued functions, denoted by $\mathrm{Sym}^0_m(\boldsymbol{A})$ henceforth, is an equivalent form of the minimal generating set given in Xiao (1996*d*).

In the following, we prove that the set given in the angle brackets in the above table, denoted by $I^0_m(\boldsymbol{A})$, provides an irreducible functional basis for scalar-valued anisotropic functions of the variable $\boldsymbol{A}\in\mathrm{Sym}$ under $C_{2mh}(\boldsymbol{n})$. The proof for $I^0_1(\boldsymbol{A})$ is already available (see Xiao 1996*d*). Thus we only need to consider the set $I^0_m(\boldsymbol{A})$ for $m\geqslant 2$.

Let $\boldsymbol{e}_1\equiv\boldsymbol{e}$ and $\boldsymbol{e}_2$ be two orthonormal vectors in the $\boldsymbol{n}$-plane, the former defining the vectors $\boldsymbol{l}_r$ and $\boldsymbol{l}'_r$ in (4.3). Denoting $\boldsymbol{e}_3\equiv\boldsymbol{n}$ and

$$\left.\begin{aligned} x = \tfrac{1}{2}(A_{11}-A_{22}), &\qquad y = A_{12}, \\ \boldsymbol{q}(\boldsymbol{A}) &= x\boldsymbol{e}_1 + y\boldsymbol{e}_2, \\ A_{ij} = \boldsymbol{e}_i\cdot\boldsymbol{A}\boldsymbol{e}_j, &\qquad i,j = 1,2,3, \end{aligned}\right\} \tag{4.10}$$

for any unit vector $\boldsymbol{l}$ in the $\boldsymbol{n}$-plane we have

$$\boldsymbol{l}\cdot\boldsymbol{A}\boldsymbol{l} = \tfrac{1}{2}(A_{11}+A_{22}) + x\cos 2\gamma + y\sin 2\gamma, \tag{4.11}$$

where γ is the angle between $\boldsymbol{l}$ and $\boldsymbol{e}$ (=$\boldsymbol{e}_1$). In particular, when $\boldsymbol{l}=\boldsymbol{l}_r, \boldsymbol{l}'_r$ (cf. (4.3)), the above identity yields

$$\boldsymbol{l}_r\cdot\mathring{\boldsymbol{A}}\boldsymbol{l}_r = x\cos\frac{2r\pi}{m} + y\sin\frac{2r\pi}{m} = |\boldsymbol{q}(\boldsymbol{A})|\cos\left(\frac{2r\pi}{m}-\theta_1\right),$$
$$\boldsymbol{l}'_r\cdot\mathring{\boldsymbol{A}}\boldsymbol{l}'_r = x\cos 2\left(\frac{r\pi}{m}+\frac{\pi}{4m}\right) + y\sin 2\left(\frac{r\pi}{m}+\frac{\pi}{4m}\right) = |\boldsymbol{q}(\boldsymbol{A})|\cos\left(\frac{2r\pi}{m}+\frac{\pi}{2m}-\theta_1\right),$$

where θ_1 is the angle between the vectors $\boldsymbol{q}(\boldsymbol{A})$ and $\boldsymbol{e}$. Hence, we have

$$\beta_m(\mathring{\boldsymbol{A}}) = |\boldsymbol{q}(\boldsymbol{A})|^m\sum_{r=1}^{m}\cos^m\left(\frac{2r\pi}{m}-\theta_1\right) = a_m|\boldsymbol{q}(\boldsymbol{A})|^m(b_m+\cos m\theta_1), \tag{4.12}$$

$$\beta'_m(\mathring{\boldsymbol{A}}) = |\boldsymbol{q}(\boldsymbol{A})|^m\sum_{r=1}^{m}\cos^m\left(\frac{2r\pi}{m}+\frac{\pi}{2m}-\theta_1\right) = a_m|\boldsymbol{q}(\boldsymbol{A})|^m(b_m+\sin m\theta_1). \tag{4.13}$$

On the other hand, we have

$$\alpha_m(\mathring{\boldsymbol{A}}\boldsymbol{n}) = |\mathring{\boldsymbol{A}}\boldsymbol{n}|^{2m}\sum_{r=1}^{m}\cos^{2m}\left(\frac{r\pi}{m}-\theta_2\right) = c_m|\mathring{\boldsymbol{A}}\boldsymbol{n}|^{2m}(d_m+\cos 2m\theta_2), \tag{4.14}$$

$$\alpha'_m(\mathring{\boldsymbol{A}}\boldsymbol{n}) = |\mathring{\boldsymbol{A}}\boldsymbol{n}|^{2m} \sum_{r=1}^{m} \cos^{2m}\left(\frac{r\pi}{m} + \frac{\pi}{4m} - \theta_2\right) = c_m |\mathring{\boldsymbol{A}}\boldsymbol{n}|^{2m}(d_m + \sin 2m\theta_2), \tag{4.15}$$

where θ_2 is the angle between the vectors $\mathring{\boldsymbol{A}}\boldsymbol{n}$ and $\boldsymbol{e}$. In the above four expressions, $a_m, \ldots, d_m$ are four constants with $a_m c_m \neq 0$.

From (4.12)–(4.15), we know that the last four invariants listed in the table of this subsection can determine the invariants $|\mathring{\boldsymbol{A}}\boldsymbol{n}|^2$ and $|\boldsymbol{q}(\boldsymbol{A})|^2$. Therefore, the four invariants given by (4.12)–(4.15), together with the two invariants $A_{33} = \boldsymbol{n} \cdot \boldsymbol{A}\boldsymbol{n}$ and $\operatorname{tr} \boldsymbol{A}$, can determine the two invariants

$$\boldsymbol{n} \cdot \boldsymbol{A}^2\boldsymbol{n} = A_{33}^2 + |\mathring{\boldsymbol{A}}\boldsymbol{n}|^2, \qquad \operatorname{tr} \boldsymbol{A}^2 = \tfrac{1}{2}(\operatorname{tr} \boldsymbol{A} - A_{33})^2 + A_{33}^2 + 2(|\mathring{\boldsymbol{A}}\boldsymbol{n}|^2 + |\boldsymbol{q}(\boldsymbol{A})|^2).$$

Applying this fact and the fact that the set

$$I_\infty(\boldsymbol{A}) = \{\boldsymbol{n} \cdot \boldsymbol{A}\boldsymbol{n}, \operatorname{tr} \boldsymbol{A}, \operatorname{tr} \boldsymbol{A}^2, \boldsymbol{n} \cdot \boldsymbol{A}^2\boldsymbol{n}, \operatorname{tr} \boldsymbol{A}^3, [\boldsymbol{n}, \boldsymbol{A}\boldsymbol{n}, \boldsymbol{A}^2\boldsymbol{n}]\}$$

is a functional basis of the variable $\boldsymbol{A} \in \mathrm{Sym}$ under the transverse isotropy group $C_{\infty h}$ (see Xiao 1996*d*, pp. 26–27), we infer

$$I_m^0(\bar{\boldsymbol{A}}) = I_m^0(\boldsymbol{A}) \Longrightarrow I_\infty(\bar{\boldsymbol{A}}) = I_\infty(\boldsymbol{A}) \Longrightarrow \exists \boldsymbol{Q} = \boldsymbol{R}_{\boldsymbol{n}}^{\psi} \in C_{\infty h}(\boldsymbol{n}) : \bar{\boldsymbol{A}} = \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^{\mathrm{T}}.$$

Moreover, if $\mathring{\boldsymbol{A}} \neq \boldsymbol{O}$, i.e.

$$|\mathring{\boldsymbol{A}}\boldsymbol{n}|^2 + |\boldsymbol{q}(\boldsymbol{A})|^2 \neq 0,$$

then utilizing the result just derived and the expressions (4.12)–(4.15), as well as the formulae $(1.14)_1$–$(1.15)_1$ in Xiao (1998*a*), we further deduce that either

$$I_m^0(\bar{\boldsymbol{A}}) = I_m^0(\boldsymbol{A}) \Longrightarrow \begin{cases} \alpha_m(\mathring{\bar{\boldsymbol{A}}}\boldsymbol{n}) = \alpha_m(\mathring{\boldsymbol{A}}\boldsymbol{n}), \\ \alpha'_m(\mathring{\bar{\boldsymbol{A}}}\boldsymbol{n}) = \alpha'_m(\mathring{\boldsymbol{A}}\boldsymbol{n}), \end{cases}$$
$$\Longrightarrow \begin{cases} \cos 2m(\theta_2 + \psi) = \cos 2m\theta_2, \\ \sin 2m(\theta_2 + \psi) = \sin 2m\theta_2, \end{cases} \Longrightarrow 2m\psi = 2k\pi,$$

or

$$\begin{cases} \beta_m(\mathring{\bar{\boldsymbol{A}}}) = \beta_m(\mathring{\boldsymbol{A}}), \\ \beta'_m(\mathring{\bar{\boldsymbol{A}}}) = \beta'_m(\mathring{\boldsymbol{A}}), \end{cases} \Longrightarrow \begin{cases} \cos m(\theta_1 + 2\psi) = \cos m\theta_1, \\ \sin m(\theta_1 + 2\psi) = \sin m\theta_1, \end{cases} \Longrightarrow 2m\psi = 2k\pi,$$

holds, i.e. $\boldsymbol{Q} \in C_{2mh}(\boldsymbol{n})$. Hence, we know that the set $I_m^0(\boldsymbol{A})$ for $m \geqslant 2$ obeys the criterion (1.5) if

$$\mathring{\boldsymbol{A}} \neq \boldsymbol{O}.$$

In addition, it is evident that the set $I_m^0(\boldsymbol{A})$ also obeys (1.5) if $\mathring{\boldsymbol{A}} = \boldsymbol{O}$. Thus, we conclude that the set $I_m^0(\boldsymbol{A})$ is a functional basis of the variable $\boldsymbol{A} \in \mathrm{Sym}$ under $C_{2mh}(\boldsymbol{n})$. The irreducibility of this basis can be inferred by the following eight pairs $X = \boldsymbol{A}$ and $X = \boldsymbol{A}'$ meeting the condition (1.6).

$\operatorname{tr} \boldsymbol{A}$:
$\boldsymbol{A} = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 - 2\boldsymbol{n} \vee \boldsymbol{e}_2, \boldsymbol{A}' = -\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + 2\boldsymbol{n} \vee \boldsymbol{e}_2;$
$\operatorname{tr} \boldsymbol{A}^3$:
$\boldsymbol{A} = \boldsymbol{e}_1 \vee \boldsymbol{e}_2 + \boldsymbol{n} \otimes \boldsymbol{e}_1 + \boldsymbol{n} \vee \boldsymbol{e}_2, \boldsymbol{A}' = -\boldsymbol{e}_1 \vee \boldsymbol{e}_2 + \boldsymbol{n} \otimes \boldsymbol{e}_1 + \boldsymbol{n} \vee \boldsymbol{e}_2;$
$\boldsymbol{n} \cdot \boldsymbol{A}\boldsymbol{n}$:
$\boldsymbol{A} = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \boldsymbol{n} \otimes \boldsymbol{n}, \boldsymbol{A}' = -\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{n} \otimes \boldsymbol{n};$

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$[\boldsymbol{n}, \boldsymbol{An}, \boldsymbol{A}^2\boldsymbol{n}]$:
$\boldsymbol{A} = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{n} \vee \boldsymbol{e}_1 + \boldsymbol{n} \vee \boldsymbol{e}_2, \boldsymbol{A}' = -\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{n} \vee \boldsymbol{e}_1 + \boldsymbol{n} \vee \boldsymbol{e}_2;$
$\alpha_m(\mathring{\boldsymbol{A}}\boldsymbol{n})$:
$\boldsymbol{A} = \boldsymbol{n} \vee \boldsymbol{e}_1, \boldsymbol{A}' = \boldsymbol{n} \vee (\boldsymbol{Q}\boldsymbol{e}_1), \boldsymbol{Q} = \boldsymbol{R}_{\boldsymbol{n}}^{\pi/2m};$
$\alpha'_m(\boldsymbol{A})$:
$\boldsymbol{A} = \boldsymbol{n} \vee (\boldsymbol{Q}\boldsymbol{e}_1), \boldsymbol{A}' = -\boldsymbol{n} \vee (\boldsymbol{Q}\boldsymbol{e}_1), \boldsymbol{Q} = \boldsymbol{R}_{\boldsymbol{n}}^{\pi/4m};$
$\beta_m(\mathring{\boldsymbol{A}})$:
$\boldsymbol{A} = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_2, \boldsymbol{A}' = \boldsymbol{Q}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_2)\boldsymbol{Q}^{\mathrm{T}}, \boldsymbol{Q} = \boldsymbol{R}_{\boldsymbol{n}}^{\pi/m};$
$\beta'_m(\mathring{\boldsymbol{A}})$:
$\boldsymbol{A} = \boldsymbol{Q}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_2)\boldsymbol{Q}^{\mathrm{T}}, \boldsymbol{A}' = -\boldsymbol{Q}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_2)\boldsymbol{Q}^{\mathrm{T}}, \boldsymbol{Q} = \boldsymbol{R}_{\boldsymbol{n}}^{\pi/2m}.$

Case 4. $\exists \boldsymbol{u}, \boldsymbol{v} \in X : g(X) \cap C_{2mh}(\boldsymbol{n}) = g(\boldsymbol{u}, \boldsymbol{v}) \cap C_{2mh}(\boldsymbol{n}) \neq g(\boldsymbol{x}) \cap C_{2mh}(\boldsymbol{n}), \boldsymbol{x} = \boldsymbol{u}, \boldsymbol{v}$.

It is evident that

$$g(\boldsymbol{x}) \cap C_{2mh}(\boldsymbol{n}) \neq C_1, \qquad \boldsymbol{x} = \boldsymbol{u}, \boldsymbol{v}.$$

Hence, from (4.1) we infer

$$(\boldsymbol{x} \cdot \boldsymbol{n})\boldsymbol{x} \times \boldsymbol{n} = \boldsymbol{0}, \qquad \boldsymbol{x} = \boldsymbol{u}, \boldsymbol{v}.$$

From the latter and the conditions given at the outset, we obtain

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ or } \boldsymbol{v} \cdot \boldsymbol{n} = 0, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \tag{4.16}$$

with $|\boldsymbol{u}| \cdot |\boldsymbol{v}| \neq 0$ and

$$g(\boldsymbol{u}, \boldsymbol{v}) \cap C_{2mh}(\boldsymbol{n}) = C_1.$$

Accordingly, we construct the following table (the first case in (4.16) is considered):

V	$\mathring{\boldsymbol{u}}, \mathring{\boldsymbol{u}} \times \boldsymbol{n}, (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n},$
Skw	$\boldsymbol{N}, \boldsymbol{u} \wedge \boldsymbol{v}, (\boldsymbol{u} \times \boldsymbol{n}) \wedge \boldsymbol{v} + (\boldsymbol{v} \times \boldsymbol{n}) \wedge \boldsymbol{u},$
Sym	$\boldsymbol{n} \otimes \boldsymbol{n}; \delta_{1m}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1, \delta_{1m}\boldsymbol{e}_2 \otimes \boldsymbol{e}_2, \delta_{1m}\boldsymbol{e}_1 \vee \boldsymbol{e}_2; (1 - \delta_{1m})\boldsymbol{I},$ $(1 - \delta_{1m})\mathring{\boldsymbol{u}} \otimes \mathring{\boldsymbol{u}}, (1 - \delta_{1m})\mathring{\boldsymbol{u}} \vee (\mathring{\boldsymbol{u}} \times \boldsymbol{n}); \boldsymbol{u} \vee \boldsymbol{v}, (\boldsymbol{u} \times \boldsymbol{n}) \vee \boldsymbol{v} + (\boldsymbol{v} \times \boldsymbol{n}) \vee \boldsymbol{u},$
R	$\mathring{\boldsymbol{u}} \cdot \mathring{\boldsymbol{r}}, (\boldsymbol{v} \cdot \boldsymbol{n})(\boldsymbol{r} \cdot \boldsymbol{n}), [\boldsymbol{n}, \mathring{\boldsymbol{u}}, \mathring{\boldsymbol{r}}],$ $\operatorname{tr} \boldsymbol{HN}, \boldsymbol{u} \cdot \boldsymbol{Hv}, [\boldsymbol{n}, \boldsymbol{u}, \boldsymbol{Hv}] + [\boldsymbol{n}, \boldsymbol{v}, \boldsymbol{Hu}],$ $\boldsymbol{n} \cdot \boldsymbol{Cn}; \delta_{1m}\boldsymbol{e}_1 \cdot \boldsymbol{Ce}_1, \delta_{1m}\boldsymbol{e}_2 \cdot \boldsymbol{Ce}_2, \delta_{1m}\boldsymbol{e}_1 \cdot \boldsymbol{Ce}_2; (1 - \delta_{1m}) \operatorname{tr} \boldsymbol{C},$ $(1 - \delta_{1m})\mathring{\boldsymbol{u}} \cdot \boldsymbol{C}\mathring{\boldsymbol{u}}, (1 - \delta_{1m})[\boldsymbol{n}, \mathring{\boldsymbol{u}}, \boldsymbol{C}\mathring{\boldsymbol{u}}]; \boldsymbol{u} \cdot \boldsymbol{Cv}, [\boldsymbol{n}, \boldsymbol{u}, \boldsymbol{Cv}] + [\boldsymbol{n}, \boldsymbol{v}, \boldsymbol{Cu}]$ $\langle(\boldsymbol{v} \cdot \boldsymbol{n})^2; \delta_{1m}(\boldsymbol{u} \cdot \boldsymbol{e}_1)^2, \delta_{1m}(\boldsymbol{u} \cdot \boldsymbol{e}_2)^2, \delta_{1m}(\boldsymbol{u} \cdot \boldsymbol{e}_1)(\boldsymbol{u} \cdot \boldsymbol{e}_2);$ $(1 - \delta_{1m})\alpha_m(\mathring{\boldsymbol{u}}), (1 - \delta_{1m})\alpha'_m(\mathring{\boldsymbol{u}})\rangle.$

By virtue of (4.16), the above table for irreducible representations can easily be constructed.

Case 5. $\exists (\boldsymbol{u}, \boldsymbol{W}) \in X : g(X) \cap C_{2mh}(\boldsymbol{n}) = g(\boldsymbol{u}, \boldsymbol{W}) \cap C_{2mh}(\boldsymbol{n}) \neq g(\boldsymbol{x}) \cap C_{2mh}(\boldsymbol{n})$, $\boldsymbol{x} = \boldsymbol{u}, \boldsymbol{W}$.

It is evident that

$$\left.\begin{array}{r} g(\boldsymbol{u}) \cap C_{2mh}(\boldsymbol{n}) \neq C_1, \quad \boldsymbol{u} \neq \boldsymbol{0}, \\ g(\boldsymbol{W}) \cap C_{2mh}(\boldsymbol{n}) \neq C_{2mh}(\boldsymbol{n}). \end{array}\right\}$$

Then, from (4.1) and (4.4) we derive

$$\left.\begin{aligned}(\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{0},\quad \boldsymbol{u}\neq\boldsymbol{0},\\ g(\boldsymbol{W})\cap C_{2mh}(\boldsymbol{n})=S_2,\quad \text{i.e. } \boldsymbol{W}\boldsymbol{n}\neq\boldsymbol{0},\end{aligned}\right\} \tag{4.17}$$

with

$$g(\boldsymbol{u},\boldsymbol{W})\cap C_{2mh}(\boldsymbol{n})=C_1.$$

For the case at issue, owing to $(4.17)_2$, irreducible generating sets for skewsymmetric and symmetric tensor-valued functions of the variables $(\boldsymbol{u},\boldsymbol{W})$ under $C_{2mh}(\boldsymbol{n})$ are provided by those for skewsymmetric and symmetric tensor-valued functions of $\boldsymbol{W}$ under $C_{2mh}(\boldsymbol{n})$, listed by the table in Case 2. Therefore, it suffices to consider scalar- and vector-valued functions and the relevant invariants only. The other invariants can be found in the table for Case 2.

V $\quad \mathring{\boldsymbol{u}}, (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}, \mathring{\boldsymbol{u}}\times\boldsymbol{n}, \boldsymbol{W}\boldsymbol{u}, \boldsymbol{W}(\boldsymbol{u}\times\boldsymbol{n})+\boldsymbol{n}\times\boldsymbol{W}\boldsymbol{u},$

R $\quad \mathring{\boldsymbol{u}}\cdot\mathring{\boldsymbol{r}}, (\boldsymbol{u}\cdot\boldsymbol{n})(\boldsymbol{r}\cdot\boldsymbol{n}), [\boldsymbol{n},\mathring{\boldsymbol{u}},\mathring{\boldsymbol{r}}], \boldsymbol{r}\cdot\boldsymbol{W}\boldsymbol{u}, [\boldsymbol{n},\boldsymbol{u},\boldsymbol{W}\boldsymbol{r}]+[\boldsymbol{n},\boldsymbol{r},\boldsymbol{W}\boldsymbol{u}]$

$\quad \langle I_m^0(\boldsymbol{u}), I_m^0(\boldsymbol{W}), (1-\delta_{1m})(\boldsymbol{u}\cdot\boldsymbol{W}\boldsymbol{n})^2, (1-\delta_{1m})(\boldsymbol{u}\cdot\boldsymbol{W}\boldsymbol{n})[\boldsymbol{u},\boldsymbol{W}\boldsymbol{n},\boldsymbol{n}]\rangle.$

We are in a position to prove that the above two sets, denoted by $V(\boldsymbol{u},\boldsymbol{W})$ and $I(\boldsymbol{u},\boldsymbol{W})$, are an irreducible generating set and a functional basis for the variables $(\boldsymbol{u},\boldsymbol{W})$ specified by (4.17) relative to $C_{2mh}(\boldsymbol{n})$, respectively. For the case when $\boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{0}$, the proof is easy. In the following, we consider the case when $\boldsymbol{u}\cdot\boldsymbol{n}=0$. We prove that the two sets in question obey the criteria (1.1) and (1.5), respectively. First, we have

$$\operatorname{rank} V(\boldsymbol{u},\boldsymbol{W})=\begin{cases}\operatorname{rank}\{\boldsymbol{u},\boldsymbol{u}\times\boldsymbol{n},\boldsymbol{W}\boldsymbol{u}\}=3, & \boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{u}\neq 0,\\ \operatorname{rank}\{\boldsymbol{u},\boldsymbol{u}\times\boldsymbol{n},\boldsymbol{W}(\boldsymbol{u}\times\boldsymbol{n})\}=3, & \boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{u}=0,\end{cases}$$

where in the second step, we have

$$\boldsymbol{n}\cdot\boldsymbol{W}(\boldsymbol{u}\times\boldsymbol{n})=(\boldsymbol{W}\boldsymbol{n})\cdot(\boldsymbol{u}\times\boldsymbol{n})\neq 0,$$

which holds for the following facts: (1) the vectors $\boldsymbol{u},\boldsymbol{u}\times\boldsymbol{n}$ and $\boldsymbol{W}\boldsymbol{n}$ lie in the $\boldsymbol{n}$-plane and the first two are independent, and (2) $\boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{u}=(\boldsymbol{W}\boldsymbol{n})\cdot\boldsymbol{u}=0$. Then, from the above and (1.2) we know that the set $V(\boldsymbol{u},\boldsymbol{W})$ obeys the criterion (1.1) for the case $\boldsymbol{u}\cdot\boldsymbol{n}=0$.

Hence, the set $V(\boldsymbol{u},\boldsymbol{W})$ is a generating set required. Let $\boldsymbol{u}=\boldsymbol{n}$ and $\boldsymbol{W}=\boldsymbol{n}\wedge(\boldsymbol{e}_1+\boldsymbol{e}_2)$, which meet (4.17). Then, we have $\mathring{\boldsymbol{u}}=\mathring{\boldsymbol{u}}\times\boldsymbol{n}=\boldsymbol{0}$, and hence we deduce that each of the three generators $(\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}$, $\boldsymbol{W}\boldsymbol{u}$ and $\boldsymbol{W}(\boldsymbol{u}\times\boldsymbol{n})+\boldsymbol{n}\times\boldsymbol{W}\boldsymbol{u}$ cannot be removed from the set $V(\boldsymbol{u},\boldsymbol{W})$. On the other hand, let $\boldsymbol{u}=\boldsymbol{e}_1$ and $\boldsymbol{W}=\boldsymbol{n}\wedge\boldsymbol{e}_2$. Then, we have: $(\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}=\boldsymbol{W}\boldsymbol{u}=\boldsymbol{0}$, and hence we infer that either of the two generators $\mathring{\boldsymbol{u}}$ and $\mathring{\boldsymbol{u}}\times\boldsymbol{n}$ cannot be removed from the set $V(\boldsymbol{u},\boldsymbol{W})$. From these facts, we conclude that the generating set $V(\boldsymbol{u},\boldsymbol{W})$ is irreducible.

Next, for $m=1$, it is readily verified that the set $I(\boldsymbol{u},\boldsymbol{W})$ obeys the criterion (1.5). For $m\geqslant 2$, let $I'(\boldsymbol{u},\boldsymbol{W})=I_m^0(\boldsymbol{u})\cup I_m^0(\boldsymbol{W})$. Then, for the pair $(\bar{\boldsymbol{u}},\bar{\boldsymbol{W}})$ and $(\boldsymbol{u},\boldsymbol{W})$, where the two vectors lie in the $\boldsymbol{n}$-plane, we have

$$I'(\bar{\boldsymbol{u}},\bar{\boldsymbol{W}})=I'(\boldsymbol{u},\boldsymbol{W})\Longrightarrow\exists\boldsymbol{R},\boldsymbol{Q}\in C_{2mh}(\boldsymbol{n}):\bar{\boldsymbol{u}}=\boldsymbol{R}\boldsymbol{u},\bar{\boldsymbol{W}}=\boldsymbol{Q}\boldsymbol{W}\boldsymbol{Q}^{\mathrm{T}}.$$

In the above, the following fact is used: $I_m^0(\boldsymbol{u})$ and $I_m^0(\boldsymbol{W})$ are, respectively, functional bases of $\boldsymbol{u}$ and $\boldsymbol{W}$ under $C_{2mh}(\boldsymbol{n})$. Denoting $\boldsymbol{Q}_0=\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{R}=\pm\boldsymbol{R}_{\boldsymbol{n}}^{\psi}$, we infer

$$(\bar{\boldsymbol{u}}\cdot\bar{\boldsymbol{W}}\boldsymbol{n})^2=((\boldsymbol{Q}_0\boldsymbol{u})\cdot\boldsymbol{W}\boldsymbol{n})^2=|\boldsymbol{u}|^2|\boldsymbol{W}\boldsymbol{n}|^2\cos^2(\theta+\delta\psi),\delta^2=1,$$

$$(\mathring{\bar{\boldsymbol{u}}} \cdot \bar{\boldsymbol{W}}\boldsymbol{n})[\mathring{\bar{\boldsymbol{u}}}, \bar{\boldsymbol{W}}\boldsymbol{n}, \boldsymbol{n}] = ((\boldsymbol{Q}_0\boldsymbol{u}) \cdot \boldsymbol{W}\boldsymbol{n})[\boldsymbol{Q}_0\boldsymbol{u}, \boldsymbol{W}\boldsymbol{n}, \boldsymbol{n}] \\ = |\boldsymbol{u}|^2|\boldsymbol{W}\boldsymbol{n}|^2\cos(\theta+\delta\psi)\sin(\theta+\delta\psi),$$

where θ is the angle between $\boldsymbol{u}(=\mathring{\boldsymbol{u}})$ and $\boldsymbol{W}\boldsymbol{n}$ and $(\theta+\delta\psi)$ the angle between $\boldsymbol{R}_{\boldsymbol{n}}^{\psi}\boldsymbol{u}$ and $\boldsymbol{W}\boldsymbol{n}$. Thus, we deduce

$$\begin{cases}(\bar{\boldsymbol{u}} \cdot \bar{\boldsymbol{W}}\boldsymbol{n})^2 = (\boldsymbol{u} \cdot \boldsymbol{W}\boldsymbol{n})^2, \\ (\bar{\boldsymbol{u}} \cdot \bar{\boldsymbol{W}}\boldsymbol{n})[\bar{\boldsymbol{u}}, \bar{\boldsymbol{W}}\boldsymbol{n}, \boldsymbol{n}] = (\boldsymbol{u} \cdot \boldsymbol{W}\boldsymbol{n})[\boldsymbol{u}, \boldsymbol{W}\boldsymbol{n}, \boldsymbol{n}],\end{cases} \Longrightarrow \begin{cases}\cos 2(\theta+\delta\psi) = \cos 2\theta, \\ \sin 2(\theta+\delta\psi) = \sin 2\theta,\end{cases}$$

Therefore, $\psi = k\pi$, i.e.

$$\boldsymbol{R} = \boldsymbol{Q}\boldsymbol{Q}_0 = \epsilon\boldsymbol{Q}\boldsymbol{R}_{\boldsymbol{n}}^{k\pi}, \qquad \epsilon^2 = 1.$$

Let $\boldsymbol{R}_0 = (-1)^k\epsilon\boldsymbol{Q} \in C_{2mh}(\boldsymbol{n})$. Then, using the fact

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \Longrightarrow \boldsymbol{R}_{\boldsymbol{n}}^{k\pi}\boldsymbol{u} = (-1)^k\boldsymbol{u},$$

we infer

$$\boldsymbol{R}_0\boldsymbol{u} = (-1)^k\epsilon\boldsymbol{Q}\boldsymbol{u} = \epsilon\boldsymbol{Q}\boldsymbol{R}_{\boldsymbol{n}}^{k\pi}\boldsymbol{u} = \boldsymbol{R}\boldsymbol{u} = \bar{\boldsymbol{u}}, \\ \boldsymbol{R}_0\boldsymbol{W}\boldsymbol{R}_0^{\mathrm{T}} = \boldsymbol{Q}\boldsymbol{W}\boldsymbol{Q}^{\mathrm{T}} = \bar{\boldsymbol{W}}.$$

Thus, we conclude that the set $I(\boldsymbol{u}, \boldsymbol{W})$ obeys the criterion (1.5) for the case at issue, and hence it is a functional basis required.

Case 6. $\exists(\boldsymbol{u}, \boldsymbol{A}) \in X : g(X) \cap C_{2mh}(\boldsymbol{n}) = g(\boldsymbol{u}, \boldsymbol{A}) \cap C_{2mh}(\boldsymbol{n}) \neq g(\boldsymbol{x}) \cap C_{2mh}(\boldsymbol{n})$, $\boldsymbol{x} = \boldsymbol{u}, \boldsymbol{A}$.

From the above conditions we derive (cf. Case 5)

$$\left.\begin{array}{r}(\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}, \qquad \boldsymbol{u} \neq \boldsymbol{0}, \\ g(\boldsymbol{A}) \cap C_{2mh}(\boldsymbol{n}) = S_2, \quad \text{i.e. } \boldsymbol{n} \times \boldsymbol{A}\boldsymbol{n} \neq \boldsymbol{0},\end{array}\right\} \tag{4.18}$$

with

$$g(\boldsymbol{u}, \boldsymbol{A}) \cap C_{2mh}(\boldsymbol{n}) = C_1.$$

For the case at issue, owing to $(4.18)_2$, irreducible generating sets for skewsymmetric and symmetric tensor-valued functions of the variables $(\boldsymbol{u}, \boldsymbol{A})$ under $C_{2mh}(\boldsymbol{n})$ are provided by those for skewsymmetric and symmetric tensor-valued functions of $\boldsymbol{A}$ under $C_{2mh}(\boldsymbol{n})$, listed by the table in Case 3. Therefore, it suffices to consider scalar- and vector-valued functions and the relevant invariants only. The other invariants can be found in the table for Case 3:

V	$\mathring{\boldsymbol{u}}, (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n}, \mathring{\boldsymbol{u}} \times \boldsymbol{n}, \boldsymbol{A}\boldsymbol{u}, \boldsymbol{A}(\boldsymbol{u} \times \boldsymbol{n}) + \boldsymbol{n} \times \boldsymbol{A}\boldsymbol{u},$
R	$\mathring{\boldsymbol{u}} \cdot \mathring{\boldsymbol{r}}, (\boldsymbol{u} \cdot \boldsymbol{n})(\boldsymbol{r} \cdot \boldsymbol{n}), [\boldsymbol{n}, \mathring{\boldsymbol{u}}, \mathring{\boldsymbol{r}}], \boldsymbol{r} \cdot \boldsymbol{A}\boldsymbol{u}, [\boldsymbol{n}, \boldsymbol{u}, \boldsymbol{A}\boldsymbol{r}] + [\boldsymbol{n}, \boldsymbol{r}, \boldsymbol{A}\boldsymbol{u}]$
	$\langle I_m^0(\boldsymbol{u}), I_m^0(\boldsymbol{A}), (1-\delta_{1m})(\boldsymbol{u} \cdot \boldsymbol{A}\boldsymbol{n})^2, (1-\delta_{1m})(\boldsymbol{u} \cdot \boldsymbol{A}\boldsymbol{n})[\boldsymbol{u}, \boldsymbol{A}\boldsymbol{n}, \boldsymbol{n}]\rangle.$

It is easy to treat the case when $\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}$. Let $\boldsymbol{u} \cdot \boldsymbol{n} = 0$. Following the same procedure used in the last case, one can prove the facts: (1) the set of vector generators given is an irreducible generating set for vector-valued anisotropic functions of the variables $(\boldsymbol{u}, \boldsymbol{A})$ specified by (4.18) relative to $C_{2mh}(\boldsymbol{n})$, and (2) the set of invariants given in the angle brackets is a functional basis of the variables $(\boldsymbol{u}, \boldsymbol{A})$ specified by (4.18) relative to $C_{2mh}(\boldsymbol{n})$.

Finally, combining the above cases and applying theorem 2.4 in Xiao (1999), we arrive at the main result of this subsection as follows.

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Theorem 4.1. *The four sets given by*

$$I^0_m(\boldsymbol{u}); I^0_m(\boldsymbol{W}); I^0_m(\boldsymbol{A}); (\boldsymbol{u}\cdot\boldsymbol{n})(\boldsymbol{v}\cdot\boldsymbol{n}), \mathring{\boldsymbol{u}}\cdot\mathring{\boldsymbol{v}}, [\boldsymbol{n},\mathring{\boldsymbol{u}},\mathring{\boldsymbol{v}}];$$
$$(\boldsymbol{u}\cdot\boldsymbol{n})\mathring{\boldsymbol{u}}\cdot\boldsymbol{Wn}, (\boldsymbol{u}\cdot\boldsymbol{n})[\boldsymbol{n},\boldsymbol{Wn},\mathring{\boldsymbol{u}}], (1-\delta_{1m})(\mathring{\boldsymbol{u}}\cdot\boldsymbol{Wn})^2,$$
$$(1-\delta_{1m})(\mathring{\boldsymbol{u}}\cdot\boldsymbol{Wn})[\boldsymbol{u},\boldsymbol{Wn},\boldsymbol{n}];$$
$$(\boldsymbol{u}\cdot\boldsymbol{n})\mathring{\boldsymbol{u}}\cdot\boldsymbol{An}, (\boldsymbol{u}\cdot\boldsymbol{n})[\boldsymbol{n},\boldsymbol{An},\mathring{\boldsymbol{u}}], (1-\delta_{1m})(\mathring{\boldsymbol{u}}\cdot\boldsymbol{An})^2, (1-\delta_{1m})(\mathring{\boldsymbol{u}}\cdot\boldsymbol{An})[\boldsymbol{u},\boldsymbol{An},\boldsymbol{n}],$$
$$(1-\delta_{1m})\mathring{\boldsymbol{u}}\cdot\boldsymbol{A}\mathring{\boldsymbol{u}}, (1-\delta_{1m})[\boldsymbol{n},\mathring{\boldsymbol{u}},\boldsymbol{A}\mathring{\boldsymbol{u}}]; \boldsymbol{n}\cdot\boldsymbol{WHn}, [\boldsymbol{n},\boldsymbol{Wn},\boldsymbol{Hn}];$$
$$\boldsymbol{n}\cdot\boldsymbol{WAn}, [\boldsymbol{n},\boldsymbol{Wn},\boldsymbol{An}], (1-\delta_{1m})\boldsymbol{n}\cdot\boldsymbol{WAWn}, (1-\delta_{1m})[\boldsymbol{n},\boldsymbol{Wn},\boldsymbol{AWn}];$$
$$\mathrm{tr}\,\boldsymbol{AB}, \mathrm{tr}\,\boldsymbol{ABN}, (1-\delta_{1m})\boldsymbol{n}\cdot\mathring{\boldsymbol{A}}\boldsymbol{B}\mathring{\boldsymbol{A}}\boldsymbol{n}, (1-\delta_{1m})[\boldsymbol{n},\mathring{\boldsymbol{A}}\boldsymbol{n},\boldsymbol{B}\mathring{\boldsymbol{A}}\boldsymbol{n}];$$
$$\boldsymbol{u}\cdot\boldsymbol{Wv}, [\boldsymbol{n},\boldsymbol{u},\boldsymbol{Wv}]+[\boldsymbol{n},\boldsymbol{v},\boldsymbol{Wu}]; \boldsymbol{u}\cdot\boldsymbol{Av}, [\boldsymbol{n},\boldsymbol{u},\boldsymbol{Av}]+[\boldsymbol{n},\boldsymbol{v},\boldsymbol{Au}];$$

and

$$\mathring{\boldsymbol{u}}, \mathring{\boldsymbol{u}}\times\boldsymbol{n}, (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}; \boldsymbol{Wu}, \boldsymbol{W}(\boldsymbol{u}\times\boldsymbol{n})+\boldsymbol{n}\times\boldsymbol{Wu}; \boldsymbol{Au}, \boldsymbol{A}(\boldsymbol{u}\times\boldsymbol{n})+\boldsymbol{n}\times\boldsymbol{Au};$$

and

$$\mathrm{Skw}^0(\boldsymbol{u}); \mathrm{Skw}^0(\boldsymbol{W}); \mathrm{Skw}^0(\boldsymbol{A}); \boldsymbol{u}\wedge\boldsymbol{v}, (\boldsymbol{u}\times\boldsymbol{n})\wedge\boldsymbol{v}+(\boldsymbol{v}\times\boldsymbol{n})\wedge\boldsymbol{u};$$

and

$$\mathrm{Sym}^0_m(\boldsymbol{u}); \mathrm{Sym}^0_m(\boldsymbol{W}); \mathrm{Sym}^0_m(\boldsymbol{A}); \boldsymbol{u}\vee\boldsymbol{v}, (\boldsymbol{u}\times\boldsymbol{n})\vee\boldsymbol{v}+(\boldsymbol{v}\times\boldsymbol{n})\vee\boldsymbol{u};$$

where $\boldsymbol{u},\boldsymbol{v}=\boldsymbol{u}_1,\ldots,\boldsymbol{u}_a$; $\boldsymbol{W},\boldsymbol{H}=\boldsymbol{W}_1,\ldots,\boldsymbol{W}_b$; $\boldsymbol{A},\boldsymbol{B}=\boldsymbol{A}_1,\ldots,\boldsymbol{A}_c$; $\boldsymbol{u}\neq\boldsymbol{v}$, $\boldsymbol{W}\neq\boldsymbol{H}$, *provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables* $X\in\mathcal{D}$ *under the group* $C_{2mh}(\boldsymbol{n})$ *for each integer* $m\geqslant 1$, *respectively.*

(*b*) *The classes* C_{2m}

Let

$$I^0(\boldsymbol{H}_1,\ldots,\boldsymbol{H}_a;\boldsymbol{W}_1,\ldots,\boldsymbol{W}_b;\boldsymbol{A}_1,\ldots,\boldsymbol{A}_c),$$
$$\mathrm{Skw}^0(\boldsymbol{H}_1,\ldots,\boldsymbol{H}_a;\boldsymbol{W}_1,\ldots,\boldsymbol{W}_b;\boldsymbol{A}_1,\ldots,\boldsymbol{A}_c),$$
$$\mathrm{Sym}^0(\boldsymbol{H}_1,\ldots,\boldsymbol{H}_a;\boldsymbol{W}_1,\ldots,\boldsymbol{W}_b;\boldsymbol{A}_1,\ldots,\boldsymbol{A}_c)$$

be, respectively, an irreducible functional basis and irreducible generating sets for scalar-valued, skewsymmetric and symmetric tensor-valued anisotropic functions of $(a+b)$ skewsymmetric tensor variables and c symmetric tensor variables under a centrosymmetrical orthogonal subgroup g containing the central inversion $-\boldsymbol{I}$. Then, according to theorems 2.1–2.2 in Xiao (1996*a*), the four sets,

$$I^0(\boldsymbol{Eu}_1,\ldots,\boldsymbol{Eu}_a;\boldsymbol{W}_1,\ldots,\boldsymbol{W}_b;\boldsymbol{A}_1,\ldots,\boldsymbol{A}_c),$$
$$\boldsymbol{E}:\mathrm{Skw}^0(\boldsymbol{Eu}_1,\ldots,\boldsymbol{Eu}_a;\boldsymbol{W}_1,\ldots,\boldsymbol{W}_b;\boldsymbol{A}_1,\ldots,\boldsymbol{A}_c),$$
$$\mathrm{Skw}^0(\boldsymbol{Eu}_1,\ldots,\boldsymbol{Eu}_a;\boldsymbol{W}_1,\ldots,\boldsymbol{W}_b;\boldsymbol{A}_1,\ldots,\boldsymbol{A}_c),$$
$$\mathrm{Sym}^0(\boldsymbol{Eu}_1,\ldots,\boldsymbol{Eu}_a;\boldsymbol{W}_1,\ldots,\boldsymbol{W}_b;\boldsymbol{A}_1,\ldots,\boldsymbol{A}_c),$$

supply, respectively, an irreducible functional basis and irreducible generating sets for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of a vector variables, b skewsymmetric tensor variables and c symmetric tensor variables under the rotation subgroup of g, i.e. $g\cap\mathrm{Orth}^+$. Here, the second set above is obtained by forming the double dot product between each generator (skewsymmetric tensor-valued) and the third-order Levi-Civita or Eddington tensor $\boldsymbol{E}$. From this fact and theorem 4.1, we derive the following result.

Theorem 4.2. *The four sets given by*

$I^0_m(\boldsymbol{Eu}); I^0_m(\boldsymbol{W}); I^0_m(\boldsymbol{A}); \mathring{\boldsymbol{u}}\cdot\mathring{\boldsymbol{v}}, [\boldsymbol{n},\mathring{\boldsymbol{u}},\mathring{\boldsymbol{v}}]; [\boldsymbol{n},\boldsymbol{u},\boldsymbol{Wn}], \boldsymbol{u}\cdot\boldsymbol{Wn};$
$[\boldsymbol{n},\boldsymbol{u},\boldsymbol{An}], \boldsymbol{u}\cdot\boldsymbol{An}, (1-\delta_{1m})\boldsymbol{u}\cdot\boldsymbol{Au}, (1-\delta_{1m})[\boldsymbol{n},\boldsymbol{u},\boldsymbol{Au}]; \boldsymbol{n}\cdot\boldsymbol{WHn}, [\boldsymbol{n},\boldsymbol{Wn},\boldsymbol{Hn}];$
$\boldsymbol{n}\cdot\boldsymbol{WAn}, [\boldsymbol{n},\boldsymbol{Wn},\boldsymbol{An}], (1-\delta_{1m})\boldsymbol{n}\cdot\boldsymbol{WAWn}, (1-\delta_{1m})[\boldsymbol{n},\boldsymbol{Wn},\boldsymbol{AWn}];$
$\operatorname{tr}\boldsymbol{AB}, \operatorname{tr}\boldsymbol{ABN}, (1-\delta_{1m})\boldsymbol{n}\cdot\mathring{\boldsymbol{A}}\boldsymbol{B}\mathring{\boldsymbol{A}}\boldsymbol{n}, (1-\delta_{1m})[\boldsymbol{n},\mathring{\boldsymbol{A}}\boldsymbol{n},\boldsymbol{B}\mathring{\boldsymbol{A}}\boldsymbol{n}];$

and

$$\boldsymbol{n}, \mathring{\boldsymbol{u}}, \mathring{\boldsymbol{u}}\times\boldsymbol{n}; \boldsymbol{Wn}, \boldsymbol{n}\times\boldsymbol{Wn}; \boldsymbol{An}, \boldsymbol{n}\times\boldsymbol{An};$$

and

$$\mathrm{Skw}^0(\boldsymbol{Eu}); \mathrm{Skw}^0_m(\boldsymbol{W}); \mathrm{Skw}^0_m(\boldsymbol{A});$$

and

$$\mathrm{Sym}^0_m(\boldsymbol{Eu}); \mathrm{Sym}^0_m(\boldsymbol{W}); \mathrm{Sym}^0_m(\boldsymbol{A});$$

where $\boldsymbol{u},\boldsymbol{v}=\boldsymbol{u}_1,\dots,\boldsymbol{u}_a$; $\boldsymbol{W},\boldsymbol{H}=\boldsymbol{W}_1,\dots,\boldsymbol{W}_b$; $\boldsymbol{A},\boldsymbol{B}=\boldsymbol{A}_1,\dots,\boldsymbol{A}_c$; $\boldsymbol{u}\neq\boldsymbol{v}$, $\boldsymbol{W}\neq\boldsymbol{H}$, *provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables* $X\in\mathcal{D}$ *under the group* $C_{2m}(\boldsymbol{n})$ *for each integer* $m\geqslant 1$*, respectively.*

5. The classes S_{4m}

The classes S_{4m} include the improper tetrahedral crystal class S_4 as the particular case when $m=1$.

According to lemma 2.5 in Xiao (1999), for any given $X\in\mathcal{D}$ there exists a subset $X_0\subset X$ such that (2.1) with $g=S_{4m}(\boldsymbol{n})$ holds, where

$$X_0\in\{(\boldsymbol{u},\boldsymbol{v}),(\boldsymbol{u},\boldsymbol{W}),(\boldsymbol{u},\boldsymbol{A})\}.$$

Moreover, by using

$$g(\boldsymbol{u})\cap S_{4m}(\boldsymbol{n})=\begin{cases}S_{4m}(\boldsymbol{n}), & \boldsymbol{u}=\boldsymbol{0},\\ C_{2m}(\boldsymbol{n}), & \boldsymbol{r}\times\boldsymbol{n}=\boldsymbol{0}, \quad \boldsymbol{r}\cdot\boldsymbol{n}\neq 0,\\ C_1, & \boldsymbol{r}\times\boldsymbol{n}\neq\boldsymbol{0},\end{cases} \tag{5.1}$$

$$g(\boldsymbol{W})\cap S_{4m}(\boldsymbol{n})=\begin{cases}S_{4m}(\boldsymbol{n}), & \boldsymbol{Wn}=\boldsymbol{0},\\ C_1, & \boldsymbol{Wn}\neq\boldsymbol{0},\end{cases} \tag{5.2}$$

$$g(\boldsymbol{A})\cap S_{4m}(\boldsymbol{n})=\begin{cases}S_{4m}(\boldsymbol{n}), & \boldsymbol{A}=x\boldsymbol{I}+y\boldsymbol{n}\otimes\boldsymbol{n},\\ C_2(\boldsymbol{n}), & \boldsymbol{n}\times\boldsymbol{An}=\boldsymbol{0}, \quad \mathring{\boldsymbol{A}}\neq\boldsymbol{O},\\ C_1, & \boldsymbol{n}\times\boldsymbol{An}\neq\boldsymbol{0},\end{cases} \tag{5.3}$$

we further infer that for any given $X\in\mathcal{D}$, there exists a single vector or second-order tensor $\boldsymbol{x}\in X$ such that

$$g(X)\cap S_{4m}(\boldsymbol{n})=g(\boldsymbol{x})\cap S_{4m}(\boldsymbol{n}).$$

Accordingly, it suffices to discuss the three cases for a single variable.

Case 1. $\exists\boldsymbol{u}\in X: g(X)\cap S_{4m}(\boldsymbol{n})=g(\boldsymbol{u})\cap S_{4m}(\boldsymbol{n})$.

According to the related result in Xiao (1995*a*, 1996*a*), representations for scalar-, vector-, and second-order tensor-valued anisotropic functions of the vector variable

$\boldsymbol{u}$ under $S_{4m}(\boldsymbol{n})$ may be obtained from those for scalar-, vector-, and second-order tensor-valued isotropic functions of the extended variables $(\boldsymbol{u}, \boldsymbol{\phi}_m(\boldsymbol{u}), \boldsymbol{N})$, respectively, where

$$\boldsymbol{\phi}_m(\boldsymbol{u}) = \boldsymbol{n} \vee \boldsymbol{\rho}_m(\mathring{\boldsymbol{u}}) + \delta_{1m}(\boldsymbol{u}\cdot\boldsymbol{n})(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_2), \tag{5.4}$$

where

$$\boldsymbol{\rho}_m(\mathring{\boldsymbol{u}}) = \sum_{r=1}^{2m} (-1)^r (\mathring{\boldsymbol{u}}\cdot\boldsymbol{l}_r)^{2m-1}\boldsymbol{l}_r, \tag{5.5}$$

where the $2m$ unit vectors $\boldsymbol{l}_r$ are obtained by the replacement of m with $2m$ in $(4.3)_2$. We would mention that the second term at the right-hand side of (5.4) comes into play only when the group $S_4(\boldsymbol{n})$ is concerned. This fact implies a particular property of the group $S_4(\boldsymbol{n})$, which will be pointed out later.

Applying the above fact and the related results for isotropic functions and removing some redundant elements, we construct the following table:

V	$\boldsymbol{u}, \mathring{\boldsymbol{u}}\times\boldsymbol{n}, (\mathring{\boldsymbol{u}}\cdot\boldsymbol{\rho}_m(\mathring{\boldsymbol{u}}))\boldsymbol{n}, [\boldsymbol{n}, \mathring{\boldsymbol{u}}, \boldsymbol{\rho}_m(\mathring{\boldsymbol{u}})],$
Skw	$\boldsymbol{N}, \boldsymbol{n}\wedge\boldsymbol{\rho}_m(\mathring{\boldsymbol{u}}), \boldsymbol{n}\wedge(\boldsymbol{\rho}_m(\mathring{\boldsymbol{u}})\times\boldsymbol{n}),$
Sym	$\boldsymbol{I}, \boldsymbol{n}\otimes\boldsymbol{n}, \mathring{\boldsymbol{u}}\otimes\mathring{\boldsymbol{u}}, \mathring{\boldsymbol{u}}\vee(\mathring{\boldsymbol{u}}\times\boldsymbol{n}), \boldsymbol{\phi}_m(\boldsymbol{u}), \boldsymbol{\phi}_m(\boldsymbol{u})\boldsymbol{N} - \boldsymbol{N}\boldsymbol{\phi}_m(\boldsymbol{u}),$
R	$\boldsymbol{u}\cdot\boldsymbol{r}, [\boldsymbol{n}, \mathring{\boldsymbol{u}}, \mathring{\boldsymbol{r}}], (\mathring{\boldsymbol{u}}\cdot\boldsymbol{\rho}_m(\mathring{\boldsymbol{u}}))(\boldsymbol{r}\cdot\boldsymbol{n}), [\boldsymbol{n}, \mathring{\boldsymbol{u}}, \boldsymbol{\rho}_m(\mathring{\boldsymbol{u}})](\boldsymbol{r}\cdot\boldsymbol{n});$
	$\operatorname{tr}\boldsymbol{H}\boldsymbol{N}, \boldsymbol{n}\cdot\boldsymbol{H}\boldsymbol{\rho}_m(\mathring{\boldsymbol{u}}), [\boldsymbol{n}, \boldsymbol{W}\boldsymbol{n}, \boldsymbol{\rho}_m(\mathring{\boldsymbol{u}})];$
	$\operatorname{tr}\boldsymbol{C}, \boldsymbol{n}\cdot\boldsymbol{C}\boldsymbol{n}, \mathring{\boldsymbol{u}}\cdot\boldsymbol{C}\mathring{\boldsymbol{u}}, [\boldsymbol{n}, \mathring{\boldsymbol{u}}, \boldsymbol{C}\mathring{\boldsymbol{u}}], \operatorname{tr}\boldsymbol{\phi}_m(\boldsymbol{u})\boldsymbol{C}, \operatorname{tr}\boldsymbol{\phi}_m(\boldsymbol{u})\boldsymbol{C}\boldsymbol{N}$
	$\langle(\boldsymbol{u}\cdot\boldsymbol{n})^2, (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{u}\cdot\boldsymbol{\rho}_m(\mathring{\boldsymbol{u}}), (\boldsymbol{u}\cdot\boldsymbol{n})[\boldsymbol{n}, \mathring{\boldsymbol{u}}, \boldsymbol{\rho}_m(\mathring{\boldsymbol{u}})], \alpha_{2m}(\mathring{\boldsymbol{u}}), \alpha'_{2m}(\mathring{\boldsymbol{u}})\rangle.$

In the above, the last two invariants are obtained by replacing m with $2m$ in (4.3).

In the following, we proceed to prove that the first three sets given in the above table, denoted by $V_m^1(\boldsymbol{u})$, $\mathrm{Skw}_m^1(\boldsymbol{u})$ and $\mathrm{Sym}_m^1(\boldsymbol{u})$ henceforth, are irreducible generating sets for vector-valued and skewsymmetric and symmetric tensor-valued anisotropic functions of the vector variable $\boldsymbol{u}$ under $S_{4m}(\boldsymbol{n})$, respectively. To this end, we prove that each of these sets obeys the criterion (1.1). In fact, with help from the relevant trigonometric identity, we have

$$\begin{aligned} \boldsymbol{u}\cdot\boldsymbol{\rho}_m(\mathring{\boldsymbol{u}}) &= \sum_{r=1}^{2m}(-1)^r(\mathring{\boldsymbol{u}}\cdot\boldsymbol{l}_r)^{2m} \\ &= |\mathring{\boldsymbol{u}}|^{2m}\sum_{r=1}^{2m}(-1)^k\cos^{2m}\left(\frac{k\pi}{2m}-\theta\right) = a'_m|\mathring{\boldsymbol{u}}|^{2m}\cos 2m\theta, \end{aligned} \tag{5.6}$$

$$[\boldsymbol{n}, \boldsymbol{u}, \boldsymbol{\rho}_m(\mathring{\boldsymbol{u}})] = a'_m|\mathring{\boldsymbol{u}}|^{2m}\sin 2m\theta, \tag{5.7}$$

and (cf. (4.14)–(4.15))

$$\alpha_{2m}(\mathring{\boldsymbol{u}}) = c'_m|\mathring{\boldsymbol{u}}|^{4m}(d'_m + \cos 4m\theta), \qquad \alpha'_{2m}(\mathring{\boldsymbol{u}}) = c'_m|\mathring{\boldsymbol{u}}|^{4m}(d'_m + \sin 4m\theta), \tag{5.8}$$

where θ is the angle between $\mathring{\boldsymbol{u}}$ and $\boldsymbol{e}$, and a'_m, c'_m and d'_m are three constants with $a'_m c'_m \neq 0$.

From the first two equalities above we infer

$$\boldsymbol{u}\cdot\boldsymbol{\rho}_m(\mathring{\boldsymbol{u}}) = [\boldsymbol{n}, \mathring{\boldsymbol{u}}, \boldsymbol{\rho}_m(\mathring{\boldsymbol{u}})] = 0 \Longleftrightarrow \mathring{\boldsymbol{u}} = \boldsymbol{0}. \tag{5.9}$$

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Then, by using (5.1) and the latter, as well as (1.2)–(1.3), we deduce that $V^1_m(\boldsymbol{u})$ and $\mathrm{Skw}^1_m(\boldsymbol{u})$ obey (1.1), respectively. For the set $\mathrm{Sym}^1_m(\boldsymbol{u})$, the group $S_4(\boldsymbol{n})$ and the groups $S_{4m}(\boldsymbol{n})$ for $m \geqslant 2$ should be treated separately due to the following particular property of the former:

$$\dim \mathrm{Sym}(g(\boldsymbol{u}) \cap S_4(\boldsymbol{n})) = \begin{cases} \dim \mathrm{Sym}(S_4(\boldsymbol{n})) = 2, & \boldsymbol{u} = \boldsymbol{0}, \\ \dim \mathrm{Sym}(C_2(\boldsymbol{n})) = 4, & \mathring{\boldsymbol{u}} = \boldsymbol{0}, \quad \boldsymbol{u}\cdot\boldsymbol{n} \neq 0, \\ \dim \mathrm{Sym}(C_1) = 6, & \boldsymbol{u}\times\boldsymbol{n} \neq \boldsymbol{0}, \end{cases} \tag{5.10}$$

while for each $m \geqslant 2$,

$$\dim \mathrm{Sym}(g(\boldsymbol{u}) \cap S_{4m}(\boldsymbol{n})) = \begin{cases} 2, & \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0}, \\ 6, & \boldsymbol{u}\times\boldsymbol{n} \neq \boldsymbol{0}. \end{cases} \tag{5.11}$$

For the set $\mathrm{Sym}^1_m(\boldsymbol{u})$, with $m = 1$ and $\boldsymbol{\rho} = \boldsymbol{\rho}_1(\mathring{\boldsymbol{u}})$ we have

$$\mathrm{rank}\,\mathrm{Sym}^1_1(\boldsymbol{u}) = \begin{cases} \mathrm{rank}\{\boldsymbol{I}, \boldsymbol{n}\otimes\boldsymbol{n}\} = 2, & \boldsymbol{u} = \boldsymbol{0}, \\ \{\boldsymbol{I}, \boldsymbol{n}\otimes\boldsymbol{n}, \boldsymbol{e}_1\otimes\boldsymbol{e}_1 - \boldsymbol{e}_2\otimes\boldsymbol{e}_2, \boldsymbol{e}_1\vee\boldsymbol{e}_2\} = 4, & \mathring{\boldsymbol{u}} = \boldsymbol{0}, \\ & \boldsymbol{u} \neq \boldsymbol{0}, \\ \{\boldsymbol{I}, \boldsymbol{n}\otimes\boldsymbol{n}, \mathring{\boldsymbol{u}}\otimes\mathring{\boldsymbol{u}}, \mathring{\boldsymbol{u}}\otimes(\mathring{\boldsymbol{u}}\times\boldsymbol{n}), \boldsymbol{n}\vee\boldsymbol{\rho}, \boldsymbol{n}\vee(\boldsymbol{n}\times\boldsymbol{\rho})\} = 6, & \mathring{\boldsymbol{u}} \neq \boldsymbol{0}, \end{cases}$$

and for $m \geqslant 2$, we have

$$\mathrm{rank}\,\mathrm{Sym}^1_m(\boldsymbol{u}) = \begin{cases} \mathrm{rank}\{\boldsymbol{I}, \boldsymbol{n}\otimes\boldsymbol{n}\} = 2, & \mathring{\boldsymbol{u}} = \boldsymbol{0}, \\ 6, & \mathring{\boldsymbol{u}} \neq \boldsymbol{0}. \end{cases}$$

Thus, we infer that the set $\mathrm{Sym}^1_m(\boldsymbol{u})$ obeys (1.1) for each integer $m \geqslant 1$.

It is evident that both $\mathrm{Skw}^1_m(\boldsymbol{u})$ and $\mathrm{Sym}^1_m(\boldsymbol{u})$ are irreducible. Moreover, from the facts (cf. (5.6)–(5.7))

$$[\boldsymbol{n}, \mathring{\boldsymbol{u}}, \boldsymbol{\rho}_m(\boldsymbol{u})] = 0 \text{ for } \boldsymbol{u} = \boldsymbol{e}; \qquad \boldsymbol{u}\cdot\boldsymbol{\rho}_m(\boldsymbol{u}) = 0 \text{ for } \boldsymbol{u} = \boldsymbol{R}^{\pi/4m}_{\boldsymbol{n}}\boldsymbol{e},$$

and $g(\boldsymbol{u}) \cap S_{4m}(\boldsymbol{n}) = C_1$ for either of the two vectors given above, we deduce that $V^1_m(\boldsymbol{u})$ is irreducible.

Next, we prove that the set given in the angle brackets in the foregoing table, denoted by $I^1_m(\boldsymbol{u})$ henceforth, is an irreducible functional basis of the vector variable $\boldsymbol{u}$ under $S_{4m}(\boldsymbol{n})$. To this end, we prove that this set obeys (1.5). It is easy to treat the case when $\mathring{\boldsymbol{u}} = \boldsymbol{0}$. Now suppose $(\boldsymbol{u}\cdot\boldsymbol{n})\mathring{\boldsymbol{u}} \neq \boldsymbol{0}$. Then by using (5.6)–(5.8) we infer

$$\begin{aligned} I^1_m(\bar{\boldsymbol{u}}) = I^1_m(\boldsymbol{u}) &\Longrightarrow \begin{cases} |\mathring{\bar{\boldsymbol{u}}}| = |\mathring{\boldsymbol{u}}|, \quad \bar{\boldsymbol{u}}\cdot\boldsymbol{n} = \delta\boldsymbol{u}\cdot\boldsymbol{n}, \\ \cos 2m\bar{\theta} = \delta\cos 2m\theta, \quad \sin 2m\bar{\theta} = \delta\sin 2m\theta, \end{cases} \\ &\Longrightarrow \begin{cases} |\mathring{\bar{\boldsymbol{u}}}| = |\mathring{\boldsymbol{u}}|, \quad \bar{\boldsymbol{u}} = \delta\boldsymbol{u}\cdot\boldsymbol{n}, \\ \bar{\theta} = \theta + (4p+1-\delta)/4m, \quad p = 0, \pm 1, \ldots, \end{cases} \\ &\Longrightarrow \exists\boldsymbol{Q}\in S_{4m}(\boldsymbol{n}) : \bar{\boldsymbol{u}} = \boldsymbol{Q}\boldsymbol{u}. \end{aligned}$$

In the above, $\delta^2 = 1$; $\bar{\theta}$ and θ are the angles between $\bar{\boldsymbol{u}}$ and $\boldsymbol{e}$ and between $\boldsymbol{u}$ and $\boldsymbol{e}$, respectively; and $\boldsymbol{Q}\in S_{4m}(\boldsymbol{n})$ depends on δ, given by

$$\boldsymbol{Q} = \begin{cases} \boldsymbol{R}^{2p\pi/2m}_{\boldsymbol{n}}, & \delta = 1, \\ -\boldsymbol{R}^{(2m+2p+1)\pi/2m}_{\boldsymbol{n}}, & \delta = -1. \end{cases}$$

Suppose $\boldsymbol{u}\cdot\boldsymbol{n}=0$ and $\mathring{\boldsymbol{u}}=\boldsymbol{u}\neq\boldsymbol{0}$. Then, by using (5.8) we deduce

$$\begin{aligned} I_m^1(\bar{\boldsymbol{u}})=I_m^1(\boldsymbol{u}) &\Longrightarrow |\bar{\boldsymbol{u}}|=|\boldsymbol{u}|,\quad \cos 4m\bar\theta=\cos 4m\theta,\quad \sin 4m\bar\theta=\sin 4m\theta\\ &\Longrightarrow |\bar{\boldsymbol{u}}|=|\boldsymbol{u}|,\quad \bar\theta=\theta+\frac{k\pi}{2m},\quad k=0,\pm1,\pm2,\ldots\\ &\Longrightarrow \exists\boldsymbol{Q}\in S_{4m}(\boldsymbol{n}):\bar{\boldsymbol{u}}=\boldsymbol{Q}\boldsymbol{u}, \end{aligned}$$

where $\boldsymbol{Q}\in S_{4m}(\boldsymbol{n})$ depends on the integer k, given by

$$\boldsymbol{Q}=\begin{cases}\boldsymbol{R}_{\boldsymbol{n}}^{k\pi/2m} & \text{if } k \text{ is even},\\ -\boldsymbol{R}_{\boldsymbol{n}}^{(k+2m)\pi/2m} & \text{if } k \text{ is odd}.\end{cases}$$

From the above, we conclude that the set $I_m^1(\boldsymbol{u})$ obeys (1.5), and therefore that it is a functional basis of the vector $\boldsymbol{u}$ under $S_{4m}(\boldsymbol{n})$. Moreover, from the following pairs $(X=\boldsymbol{u},X'=\boldsymbol{u}')$, fulfilling the condition (1.6), we infer that the just-mentioned functional basis is irreducible:

$(\boldsymbol{u}\cdot\boldsymbol{n})^2$:
$\boldsymbol{u}=\boldsymbol{n},\quad \boldsymbol{u}'=2\boldsymbol{n}$;
$(\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{u}\cdot\boldsymbol{\rho}_m(\boldsymbol{u})$:
$\boldsymbol{u}=\boldsymbol{n}+\boldsymbol{e},\quad \boldsymbol{u}'=-\boldsymbol{n}+\boldsymbol{e}$;
$(\boldsymbol{u}\cdot\boldsymbol{n})[\boldsymbol{n},\mathring{\boldsymbol{u}},\boldsymbol{\rho}_m(\boldsymbol{u})]$:
$\boldsymbol{u}=\boldsymbol{n}+\boldsymbol{R}_{\boldsymbol{n}}^{\pi/4m},\quad \boldsymbol{u}'=-\boldsymbol{n}+\boldsymbol{R}_{\boldsymbol{n}}^{\pi/4m}$;
$\alpha_{2m}(\mathring{\boldsymbol{u}})$:
$\boldsymbol{u}=\boldsymbol{e},\quad \boldsymbol{u}'=\boldsymbol{R}_{\boldsymbol{n}}^{\pi/4m}\boldsymbol{e}$;
$\alpha'_{2m}(\mathring{\boldsymbol{u}})$:
$\boldsymbol{u}=\boldsymbol{R}_{\boldsymbol{n}}^{\pi/8m}\boldsymbol{e},\quad \boldsymbol{u}'=-\boldsymbol{R}_{\boldsymbol{n}}^{\pi/8m}\boldsymbol{e}$.

Case 2. $\forall\boldsymbol{r}\in X:\boldsymbol{r}\times\boldsymbol{n}=\boldsymbol{0}\,\&\,\exists\boldsymbol{W}\in X:g(X)\cap S_{4m}(\boldsymbol{n})=g(\boldsymbol{W})\cap S_{4m}(\boldsymbol{n})$.

Every scalar-valued (resp. second-order tensor-valued) anisotropic function of $\boldsymbol{W}$ under $S_{4m}(\boldsymbol{n})$ is equivalent to a scalar-valued (resp. second-order tensor-valued) anisotropic function of $\boldsymbol{W}$ under $C_{4mh}(\boldsymbol{n})$. Irreducible representations for the latter can be obtained merely replacing m with $2m$ in the table for Case 2 of § 4 *a*. In the following, we only need to consider the vector-valued function and the related invariants.

Representations for vector-valued anisotropic functions of the skewsymmetric tensor variable $\boldsymbol{W}$ under $S_{4m}(\boldsymbol{n})$ can be derived from those for isotropic functions of the extended variables $(\boldsymbol{W},\boldsymbol{\rho}_m(\boldsymbol{W}\boldsymbol{n}),\boldsymbol{N})$ (see Xiao 1995*a*, 1996*a*), where $\boldsymbol{\rho}_m(\boldsymbol{W}\boldsymbol{n})$ is defined by (5.5). Applying this fact, we derive an irreducible generating set for vector-valued anisotropic functions of the variable $\boldsymbol{W}\in$ Skw, denoted by $V_m^1(\boldsymbol{W})$ henceforth, as follows:

V	$\boldsymbol{\rho}_m(\boldsymbol{W}\boldsymbol{n}),\boldsymbol{n}\times\boldsymbol{\rho}_m(\boldsymbol{W}\boldsymbol{n}),(\boldsymbol{\rho}_m(\boldsymbol{W}\boldsymbol{n})\cdot\boldsymbol{W}\boldsymbol{n})\boldsymbol{n},[\boldsymbol{n},\boldsymbol{W}\boldsymbol{n},\boldsymbol{\rho}_m(\boldsymbol{W}\boldsymbol{n})]\boldsymbol{n}$,
R	$(\boldsymbol{r}\cdot\boldsymbol{n})(\boldsymbol{\rho}_m(\boldsymbol{W}\boldsymbol{n})\cdot\boldsymbol{W}\boldsymbol{n}),(\boldsymbol{r}\cdot\boldsymbol{n})[\boldsymbol{n},\boldsymbol{W}\boldsymbol{n},\boldsymbol{\rho}_m(\boldsymbol{W}\boldsymbol{n})]$.

The above result may readily be verified by means of the fact (cf. Case 1)

$$\boldsymbol{\rho}_m(\boldsymbol{W}\boldsymbol{n})=[\boldsymbol{n},\boldsymbol{W}\boldsymbol{n},\boldsymbol{\rho}_m(\boldsymbol{W}\boldsymbol{n})]=0\Longleftrightarrow\boldsymbol{\rho}_m(\boldsymbol{W}\boldsymbol{n})=\boldsymbol{0}\Longleftrightarrow\boldsymbol{W}\boldsymbol{n}=\boldsymbol{0}.$$

Case 3. $\forall\boldsymbol{r}\in X:\boldsymbol{r}\times\boldsymbol{n}=\boldsymbol{0}\,\&\,\exists\boldsymbol{A}\in X:g(X)\cap S_{4m}(\boldsymbol{n})=g(\boldsymbol{A})\cap S_{4m}(\boldsymbol{n})$.

Every scalar-valued (resp. second-order tensor-valued) anisotropic function of $\boldsymbol{A}$ under $S_{4m}(\boldsymbol{n})$ is equivalent to a scalar-valued (resp. second-order tensor-valued) anisotropic function of $\boldsymbol{A}$ under $C_{4mh}(\boldsymbol{n})$. Irreducible representations for the latter can be obtained merely replacing m with $2m$ in the table for Case 3 of § 4 *a*. In the following, we only need to consider the vector-valued function.

Representations for vector-valued anisotropic functions of the skewsymmetric tensor variable $\boldsymbol{A}$ under $S_{4m}(\boldsymbol{n})$ can be derived from those for isotropic functions of the extended variables

$$(\boldsymbol{A}, \boldsymbol{\rho}_m(\mathring{\boldsymbol{A}}\boldsymbol{n}), \mu_m(\mathring{\boldsymbol{A}})\boldsymbol{n}, \nu_m(\mathring{\boldsymbol{A}})\boldsymbol{n}, \boldsymbol{N})$$

(see Xiao 1995*a*, 1996*a*), where $\boldsymbol{\rho}_m(\mathring{\boldsymbol{A}}\boldsymbol{n})$ is defined by (5.5) and moreover,

$$\mu_m(\mathring{\boldsymbol{A}}) = \sum_{r=1}^{2m}(-1)^r(\boldsymbol{l}_r \cdot \mathring{\boldsymbol{A}}\boldsymbol{l}_r)^m, \qquad \nu_m(\mathring{\boldsymbol{A}}) = \sum_{r=1}^{2m}(-1)^r(\boldsymbol{l}'_r \cdot \mathring{\boldsymbol{A}}\boldsymbol{l}'_r)^m, \tag{5.12}$$

where $\boldsymbol{l}_r$ and $\boldsymbol{l}'_r$, $r = 1, 2, \ldots, 2m$, are given in (4.3) with m replaced by $2m$ therein. Applying this fact, we derive an irreducible generating set for vector-valued anisotropic functions of the variable $\boldsymbol{A} \in \mathrm{Sym}$, denoted by $V^1_m(\boldsymbol{A})$ henceforth, as follows.

V $\quad \boldsymbol{\rho}_m(\mathring{\boldsymbol{A}}\boldsymbol{n}) + \mu_m(\mathring{\boldsymbol{A}})\boldsymbol{n}, \boldsymbol{n} \times \boldsymbol{\rho}_m(\mathring{\boldsymbol{A}}\boldsymbol{n}) + \nu_m(\mathring{\boldsymbol{A}})\boldsymbol{n}, (\boldsymbol{\rho}_m(\mathring{\boldsymbol{A}}\boldsymbol{n}) \cdot \mathring{\boldsymbol{A}}\boldsymbol{n})\boldsymbol{n},$
$[\boldsymbol{n}, \mathring{\boldsymbol{A}}\boldsymbol{n}, \boldsymbol{\rho}_m(\mathring{\boldsymbol{A}}\boldsymbol{n})]\boldsymbol{n},$

R $\quad (\boldsymbol{r} \cdot \boldsymbol{n})\mu_m(\mathring{\boldsymbol{A}}), (\boldsymbol{r} \cdot \boldsymbol{n})\nu_m(\mathring{\boldsymbol{A}}), (\boldsymbol{r} \cdot \boldsymbol{n})(\boldsymbol{\rho}_m(\mathring{\boldsymbol{A}}\boldsymbol{n}) \cdot \mathring{\boldsymbol{A}}\boldsymbol{n}), (\boldsymbol{r} \cdot \boldsymbol{n})[\boldsymbol{n}, \mathring{\boldsymbol{A}}\boldsymbol{n}, \boldsymbol{\rho}_m(\mathring{\boldsymbol{A}}\boldsymbol{n})].$

In fact, by using (1.2) and the fact (cf. Case 1)

$$\boldsymbol{\rho}_m(\mathring{\boldsymbol{A}}\boldsymbol{n}) = [\boldsymbol{n}, \mathring{\boldsymbol{A}}\boldsymbol{n}, \boldsymbol{\rho}_m(\mathring{\boldsymbol{A}}\boldsymbol{n})] = 0 \Longleftrightarrow \boldsymbol{\rho}_m(\mathring{\boldsymbol{A}}\boldsymbol{n}) = \boldsymbol{0} \Longleftrightarrow \mathring{\boldsymbol{A}}\boldsymbol{n} = \boldsymbol{0},$$

as well as $(5.3)_3$, we infer that the criterion (1.1) can be satisfied when $\boldsymbol{n} \times \boldsymbol{A}\boldsymbol{n} \neq \boldsymbol{0}$, and that each of the presented generators is irreducible. On the other hand, let $\boldsymbol{n} \times \boldsymbol{A}\boldsymbol{n} = \boldsymbol{0}$. Then by using (1.2) and (5.3), we deduce that the criterion (1.1) can also be satisfied when $\boldsymbol{n} \times \boldsymbol{A}\boldsymbol{n} = \boldsymbol{0}$.

Finally, combining the above three cases, we arrive at the main result of this section as follows.

Theorem 5.1. *The four sets given by*

$I^1_m(\boldsymbol{u}); I^0_{2m}(\boldsymbol{W}); I^0_{2m}(\boldsymbol{A});$
$\boldsymbol{u} \cdot \boldsymbol{v}, [\boldsymbol{n}, \mathring{\boldsymbol{u}}, \mathring{\boldsymbol{v}}], (\boldsymbol{v} \cdot \boldsymbol{n})(\boldsymbol{u} \cdot \boldsymbol{\rho}_m(\mathring{\boldsymbol{u}})), (\boldsymbol{v} \cdot \boldsymbol{n})[\boldsymbol{n}, \mathring{\boldsymbol{u}}, \boldsymbol{\rho}_m(\mathring{\boldsymbol{u}})];$
$\boldsymbol{\rho}_m(\mathring{\boldsymbol{u}}) \cdot \boldsymbol{W}\boldsymbol{n}, [\boldsymbol{n}, \boldsymbol{W}\boldsymbol{n}, \boldsymbol{\rho}_m(\mathring{\boldsymbol{u}})], (\boldsymbol{u} \cdot \boldsymbol{n})(\boldsymbol{\rho}_m(\boldsymbol{W}\boldsymbol{n}) \cdot \boldsymbol{W}\boldsymbol{n}), (\boldsymbol{u} \cdot \boldsymbol{n})[\boldsymbol{n}, \boldsymbol{W}\boldsymbol{n}, \boldsymbol{\rho}_m(\boldsymbol{W}\boldsymbol{n})];$
$\mathring{\boldsymbol{u}} \cdot \boldsymbol{A}\mathring{\boldsymbol{u}}, [\boldsymbol{n}, \mathring{\boldsymbol{u}}, \boldsymbol{A}\mathring{\boldsymbol{u}}], \mathrm{tr}\, \boldsymbol{\phi}_m(\mathring{\boldsymbol{u}})\boldsymbol{A}, \mathrm{tr}\, \boldsymbol{\phi}_m(\mathring{\boldsymbol{u}})\boldsymbol{A}\boldsymbol{N}, (\boldsymbol{u} \cdot \boldsymbol{n})\mu_m(\mathring{\boldsymbol{A}}), (\boldsymbol{u} \cdot \boldsymbol{n})\nu_m(\mathring{\boldsymbol{A}}),$
$(\boldsymbol{u} \cdot \boldsymbol{n})(\boldsymbol{\rho}_m(\mathring{\boldsymbol{A}}\boldsymbol{n}) \cdot \mathring{\boldsymbol{A}}\boldsymbol{n}), (\boldsymbol{u} \cdot \boldsymbol{n})[\boldsymbol{n}, \mathring{\boldsymbol{A}}\boldsymbol{n}, \boldsymbol{\rho}_m(\mathring{\boldsymbol{A}}\boldsymbol{n})]; \boldsymbol{n} \cdot \boldsymbol{W}\boldsymbol{H}\boldsymbol{n}, [\boldsymbol{n}, \boldsymbol{W}\boldsymbol{n}, \boldsymbol{H}\boldsymbol{n}];$
$\boldsymbol{n} \cdot \boldsymbol{W}\boldsymbol{A}\boldsymbol{n}, [\boldsymbol{n}, \boldsymbol{W}\boldsymbol{n}, \boldsymbol{A}\boldsymbol{n}], \boldsymbol{n} \cdot \boldsymbol{W}\boldsymbol{A}\boldsymbol{W}\boldsymbol{n}, [\boldsymbol{n}, \boldsymbol{W}\boldsymbol{n}, \boldsymbol{A}\boldsymbol{W}\boldsymbol{n}];$
$\mathrm{tr}\, \boldsymbol{A}\boldsymbol{B}, \mathrm{tr}\, \boldsymbol{A}\boldsymbol{B}\boldsymbol{N}, \boldsymbol{n} \cdot \mathring{\boldsymbol{A}}\boldsymbol{B}\mathring{\boldsymbol{A}}\boldsymbol{n}, [\boldsymbol{n}, \mathring{\boldsymbol{A}}\boldsymbol{n}, \boldsymbol{B}\mathring{\boldsymbol{A}}\boldsymbol{n}];$

and

$$V^1_m(\boldsymbol{u}); V^1_m(\boldsymbol{W}); V^1_m(\boldsymbol{A});$$

and

$$\mathrm{Skw}^1_m(\boldsymbol{u}); \mathrm{Skw}^0(\boldsymbol{W}); \mathrm{Skw}^0_{2m}(\boldsymbol{A});$$

and

$$\mathrm{Sym}^1_m(\boldsymbol{u}); \mathrm{Sym}^0_{2m}(\boldsymbol{W}); \mathrm{Sym}^0_{2m}(\boldsymbol{A});$$

where $\boldsymbol{u}, \boldsymbol{v} = \boldsymbol{u}_1, \ldots, \boldsymbol{u}_a$; $\boldsymbol{W}, \boldsymbol{H} = \boldsymbol{W}_1, \ldots, \boldsymbol{W}_b$; $\boldsymbol{A}, \boldsymbol{B} = \boldsymbol{A}_1, \ldots, \boldsymbol{A}_c$; $\boldsymbol{u} \neq \boldsymbol{v}$, $\boldsymbol{W} \neq \boldsymbol{H}$, *provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables* $X \in \mathcal{D}$ *under the group* $S_{4m}(\boldsymbol{n})$ *for each integer* $m \geqslant 1$, *respectively.*

6. The classes C_{2m+1h}

The classes C_{2m+1h} include the hexagonal crystal class C_{3h} as the particular case when $m = 1$.

Applying (6.1) and (6.11) given later, we infer that for any given $\boldsymbol{u} \in V$ and $\boldsymbol{W} \in \text{Skw}$,

$$g(\boldsymbol{u}, \boldsymbol{W}) \cap C_{2m+1h}(\boldsymbol{n}) = g(\boldsymbol{x}) \cap C_{2m+1h}(\boldsymbol{n}), \boldsymbol{x} \in \{\boldsymbol{u}, \boldsymbol{W}\}.$$

This indicates that the set of two variables, $(\boldsymbol{u}, \boldsymbol{W})$, can be reduced to a single vector variable or a single skewsymmetric tensor variable. Hence in fulfilling the scheme designed in § 2, it suffices to treat the cases for a single variable and two variables: $\boldsymbol{u}$, $\boldsymbol{W}$, $\boldsymbol{A}$, $(\boldsymbol{u}, \boldsymbol{v})$ and $(\boldsymbol{u}, \boldsymbol{A})$.

Case 1. $\exists \boldsymbol{u} \in X : g(X) \cap C_{2m+1h}(\boldsymbol{n}) = g(\boldsymbol{u}) \cap C_{2m+1h}(\boldsymbol{n})$.

By means of the criterion (1.1) and (1.2)–(1.4), as well as the facts

$$g(\boldsymbol{u}) \cap C_{2m+1h}(\boldsymbol{n}) = \begin{cases} C_{2m+1h}(\boldsymbol{n}), & \boldsymbol{u} = \boldsymbol{0}, \\ C_{2m+1}(\boldsymbol{n}), & \boldsymbol{u} = a\boldsymbol{n} \neq \boldsymbol{0}, \\ C_{1h}(\boldsymbol{n}), & \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \boldsymbol{u} \neq \boldsymbol{0}, \\ C_1, & (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{u} \times \boldsymbol{n} \neq \boldsymbol{0}, \end{cases} \tag{6.1}$$

we infer that the three generating sets $V^0(\boldsymbol{u})$, $\text{Skw}^0(\boldsymbol{u})$ and $\text{Sym}^0_{2m+1}(\boldsymbol{u})$ given in the table for Case 1 of § 4 *a*, are also generating sets for vector-valued and skewsymmetric and symmetric tensor-valued anisotropic functions of the vector variable $\boldsymbol{u}$ under $C_{2m+1h}(\boldsymbol{n})$, respectively. In the following, we show that the set below is an irreducible functional basis of the vector variable $\boldsymbol{u}$ under $C_{2m+1h}(\boldsymbol{n})$.

$$I^2_m(\boldsymbol{u}) = \{(\boldsymbol{u} \cdot \boldsymbol{n}), \gamma_m(\mathring{\boldsymbol{u}}), \gamma'_m(\mathring{\boldsymbol{u}})\}, \tag{6.2}$$

where

$$\left.\begin{aligned} \gamma_m(\mathring{\boldsymbol{u}}) &= \sum_{s=1}^{2m+1} (\mathring{\boldsymbol{u}} \cdot \boldsymbol{a}_s)^{2m+1}, \qquad \boldsymbol{a}_s = \boldsymbol{R}_{\boldsymbol{n}}^{2s\pi/2m+1}\boldsymbol{e}, \\ \gamma'_m(\mathring{\boldsymbol{u}}) &= \sum_{s=1}^{2m+1} (\mathring{\boldsymbol{u}} \cdot \boldsymbol{a}'_s)^{2m+1}, \qquad \boldsymbol{a}'_s = \boldsymbol{R}_{\boldsymbol{n}}^{(4s+1)\pi/4m+2}\boldsymbol{e}, \end{aligned}\right\} \tag{6.3}$$

where $\boldsymbol{e}$ is a unit vector in the $\boldsymbol{n}$-plane. Applying the related trigonometric identity, we derive

$$\gamma_m(\mathring{\boldsymbol{u}}) = |\mathring{\boldsymbol{u}}|^{2m+1} \sum_{s=1}^{2m+1} \cos^{2m+1}\left(\frac{2s\pi}{2m+1} - \theta\right) = f_m|\mathring{\boldsymbol{u}}|^{2m+1}\cos(2m+1)\theta, \tag{6.4}$$

$$\begin{aligned} \gamma'_m(\mathring{\boldsymbol{u}}) &= |\mathring{\boldsymbol{u}}|^{2m+1} \sum_{s=1}^{2m+1} \cos^{2m+1}\left(\frac{2s\pi}{2m+1} + \frac{\pi}{4m+2} - \theta\right) \\ &= f_m|\mathring{\boldsymbol{u}}|^{2m+1}\sin(2m+1)\theta, \end{aligned} \tag{6.5}$$

where θ is the angle between $\mathring{\boldsymbol{u}}$ and $\boldsymbol{e}$ and f_m is a non-vanishing constant depending on m. With the help of the above expressions, for $\bar{\boldsymbol{u}}, \boldsymbol{u} \in V$ we deduce

$$I^2_m(\bar{\boldsymbol{u}}) = I^2_m(\boldsymbol{u}) \Longrightarrow \begin{cases} \bar{\boldsymbol{u}} \cdot \boldsymbol{n} = \delta\boldsymbol{u} \cdot \boldsymbol{n}, \quad |\mathring{\bar{\boldsymbol{u}}}| = |\mathring{\boldsymbol{u}}|, \\ \cos(2m+1)\bar{\theta} = \cos(2m+1)\theta, \\ \sin(2m+1)\bar{\theta} = \sin(2m+1)\theta, \end{cases}$$

$$\Longrightarrow \begin{cases} \bar{\boldsymbol{u}} \cdot \boldsymbol{n} = \delta \boldsymbol{u} \cdot \boldsymbol{n}, & |\mathring{\bar{\boldsymbol{u}}}| = |\mathring{\boldsymbol{u}}|, \\ \bar{\theta} = \theta + \dfrac{2k\pi}{2m+1}, & k + 0, \pm 1, \pm 2, \ldots \end{cases}$$
$$\Longrightarrow \exists \boldsymbol{Q} \in C_{2m+1h}(\boldsymbol{n}) : \bar{\boldsymbol{u}} = \boldsymbol{Q}\boldsymbol{u},$$

where $\delta^2 = 1$ and $\boldsymbol{Q} \in C_{2m+1h}(\boldsymbol{n})$ depends on δ, given by

$$\boldsymbol{Q} = \begin{cases} \boldsymbol{R}_{\boldsymbol{n}}^{2k\pi/2m+1}, & \delta = 1, \\ -\boldsymbol{R}_{\boldsymbol{n}}^{\pi}\boldsymbol{R}_{\boldsymbol{n}}^{2k\pi/2m+1}, & \delta = -1. \end{cases}$$

Thus, we infer that the set $I_m^2(\boldsymbol{u})$ obeys (1.5), and hence that it is a functional basis of $\boldsymbol{u}$ under $C_{2m+1h}(\boldsymbol{n})$. It may easily be verified that this basis is irreducible by the related method used before.

From the above, we conclude that for the case at issue, the desired results can be obtained from the table for Case 1 in § 4 *a* by taking $\delta_{1m} = 0$ and replacing the invariants $\alpha_m(\mathring{\boldsymbol{u}})$ and $\alpha'_m(\mathring{\boldsymbol{u}})$ with the invariants $\gamma_m(\mathring{\boldsymbol{u}})$ and $\gamma'_m(\mathring{\boldsymbol{u}})$.

Case 2. $\exists \boldsymbol{x} \in X : g(X) \cap C_{2m+1h}(\boldsymbol{n}) = g(\boldsymbol{x}) \cap C_{2m+1h}(\boldsymbol{n})$, where $\boldsymbol{x} \in$ Skw or $\boldsymbol{x} \in$ Sym.

Since a scalar-valued (resp. second-order tensor-valued) anisotropic function of a second-order tensor variable $\boldsymbol{x}$ under $C_{2m+1h}(\boldsymbol{n})$ is equivalent to a scalar-valued (resp. second-order tensor-valued) anisotropic function of $\boldsymbol{x}$ under $C_{4m+2h}(\boldsymbol{n})$, we know that irreducible functional bases or generating sets for scalar-valued and skew-symmetric and symmetric tensor-valued anisotropic functions of the variable $\boldsymbol{x} = \boldsymbol{W} \in$ Skw or $\boldsymbol{x} = \boldsymbol{A} \in$ Sym under $C_{2m+1h}(\boldsymbol{n})$ are provided by the corresponding ones given in the tables for Case 2 or Case 3 in § 4 *a*, with the replacement of m by $2m+1$.

Hence, in what follows, it suffices to treat the vector-valued functions of $\boldsymbol{W}$ or $\boldsymbol{A}$ and the related invariants. According to Xiao (1995*a*, 1996*a*), generating sets for vector-valued anisotropic functions of the variable $\boldsymbol{W}$ or $\boldsymbol{A}$ under $C_{2m+1h}(\boldsymbol{n})$ are obtainable from those for vector-valued isotropic functions of the extended variables

$$(\boldsymbol{W}, \boldsymbol{\xi}_m(\boldsymbol{W}\boldsymbol{n}), \boldsymbol{N}) \quad \text{or} \quad (\boldsymbol{A}, \boldsymbol{\xi}_m(\mathring{\boldsymbol{A}}\boldsymbol{n}), \boldsymbol{\eta}_m(\mathring{\boldsymbol{A}}), \boldsymbol{N}),$$

where

$$\boldsymbol{\xi}_m(\boldsymbol{x}) = \sum_{s=1}^{2m+1} (\boldsymbol{x} \cdot \boldsymbol{a}_s)^{2m} \boldsymbol{a}_s, \qquad \boldsymbol{x} \in \{\boldsymbol{W}\boldsymbol{n}, \mathring{\boldsymbol{A}}\boldsymbol{n}\}, \tag{6.6}$$

$$\boldsymbol{\eta}_m(\mathring{\boldsymbol{A}}) = \sum_{s=1}^{2m+1} (\boldsymbol{a}_s \cdot \mathring{\boldsymbol{A}}\boldsymbol{a}_s)^m \boldsymbol{a}_s. \tag{6.7}$$

In the above, the vectors $\boldsymbol{a}_s$, $s = 1, 2, \ldots, 2m+1$, have been given in $(6.3)_1$.

Taking the above facts into account, we construct the following two tables:

V	$\boldsymbol{\xi}_m(\boldsymbol{W}\boldsymbol{n}), \boldsymbol{n} \times \boldsymbol{\xi}_m(\boldsymbol{W}\boldsymbol{n}), \gamma_m(\boldsymbol{W}\boldsymbol{n})\boldsymbol{n}, \gamma'_m(\boldsymbol{W}\boldsymbol{n})\boldsymbol{n}$,
R	$\boldsymbol{r} \cdot \boldsymbol{\xi}_m(\boldsymbol{W}\boldsymbol{n}), [\boldsymbol{n}, \boldsymbol{r}, \boldsymbol{\xi}_m(\boldsymbol{W}\boldsymbol{n})], (\boldsymbol{r} \cdot \boldsymbol{n})\gamma_m(\boldsymbol{W}\boldsymbol{n}), (\boldsymbol{r} \cdot \boldsymbol{n})\gamma'_m(\boldsymbol{W}\boldsymbol{n})$,
V	$\boldsymbol{\xi}_m(\mathring{\boldsymbol{A}}\boldsymbol{n}), \boldsymbol{n} \times \boldsymbol{\xi}_m(\mathring{\boldsymbol{A}}\boldsymbol{n}), \boldsymbol{\eta}_m(\mathring{\boldsymbol{A}}) + \gamma_m(\mathring{\boldsymbol{A}})\boldsymbol{n}, \boldsymbol{n} \times \boldsymbol{\eta}_m(\mathring{\boldsymbol{A}}) + \gamma'_m(\mathring{\boldsymbol{A}}\boldsymbol{n})\boldsymbol{n}$,
R	$\boldsymbol{r} \cdot \boldsymbol{\xi}_m(\mathring{\boldsymbol{A}}\boldsymbol{n}), [\boldsymbol{n}, \boldsymbol{r}, \boldsymbol{\xi}_m(\mathring{\boldsymbol{A}}\boldsymbol{n})], (\boldsymbol{r} \cdot \boldsymbol{n})\gamma_m(\mathring{\boldsymbol{A}}\boldsymbol{n}), (\boldsymbol{r} \cdot \boldsymbol{n})\gamma'_m(\mathring{\boldsymbol{A}}\boldsymbol{n})$.

We would point out that in deriving the last two invariants in the second table, the terms

$$\boldsymbol{r}\cdot\boldsymbol{\eta}_m(\mathring{\boldsymbol{A}}) \quad \text{and} \quad [\boldsymbol{n}, \boldsymbol{r}, \boldsymbol{\eta}_m(\mathring{\boldsymbol{A}})]$$

have been removed, whereas these terms should appear when forming the inner product between the generic vector variable $\boldsymbol{r}\in X$ and the last two vector generators in the second table above. The explanation is as follows. Since the case when $(\boldsymbol{r}\cdot\boldsymbol{n})\boldsymbol{r}\times\boldsymbol{n}\neq\boldsymbol{0}$ has been covered by Case 1, we only need to consider $(\boldsymbol{r}\cdot\boldsymbol{n})\boldsymbol{r}\times\boldsymbol{n}=\boldsymbol{0}$ for each $\boldsymbol{r}\in X$. The foregoing fact is evident for the case when $\boldsymbol{r}\times\boldsymbol{n}=\boldsymbol{0}$. Now let $\boldsymbol{r}\cdot\boldsymbol{n}=0$ and $\boldsymbol{r}\neq\boldsymbol{0}$. If $\mathring{\boldsymbol{A}}\boldsymbol{n}\neq\boldsymbol{0}$, then $\boldsymbol{\xi}_m(\mathring{\boldsymbol{A}}\boldsymbol{n})$ and $\boldsymbol{n}\times\boldsymbol{\xi}_m(\mathring{\boldsymbol{A}}\boldsymbol{n})$ are two independent vectors in the $\boldsymbol{n}$-plane. From this we know that the aforementioned two terms can be determined by the first two invariants in the second table and therefore can be removed. If $\mathring{\boldsymbol{A}}\boldsymbol{n}=\boldsymbol{0}$, then we have (cf. $(6.1)_3$ and $(6.12)_{1,2}$ below)

$$\begin{aligned} g(X)\cap C_{2m+1h}(\boldsymbol{n}) &= g(\boldsymbol{A})\cap C_{2m+1h}(\boldsymbol{n})\subset C_{1h}(\boldsymbol{n})\\ &= g(\boldsymbol{r})\cap C_{2m+1h}(\boldsymbol{n})\subset g(X)\cap C_{2m+1h}(\boldsymbol{n}), \end{aligned}$$

i.e. $g(X)\cap C_{2m+1h}(\boldsymbol{n}) = g(\boldsymbol{r})\cap C_{2m+1h}(\boldsymbol{n})$, which has also been covered by Case 1.

In what follows we proceed to prove that the above two sets of vector generators given in the foregoing two tables, denoted by $I_m^2(\boldsymbol{W})$ and $I_m^2(\boldsymbol{A})$ henceforth, obey the criterion (1.1). First, we prove the following fact:

$$\left.\begin{aligned} \boldsymbol{\xi}_m(\mathring{\boldsymbol{z}})=\boldsymbol{0} &\Longleftrightarrow \mathring{\boldsymbol{z}}=\boldsymbol{0},\\ \boldsymbol{\eta}_m(\mathring{\boldsymbol{A}})=\boldsymbol{0} &\Longleftrightarrow \boldsymbol{q}(\boldsymbol{A})=\boldsymbol{0}, \end{aligned}\right\} \tag{6.8}$$

for $\boldsymbol{z}\in V$ and $\boldsymbol{A}\in\mathrm{Sym}$. In the second expression above, the vector $\boldsymbol{q}(\boldsymbol{A})$ is defined by $(4.10)_2$. In fact, using (4.11) and the relevant trigonometric identity, we derive

$$\begin{aligned} \boldsymbol{\eta}_m(\mathring{\boldsymbol{A}})\cdot\boldsymbol{l} &= |\boldsymbol{q}(\boldsymbol{A})|^m \sum_{s=1}^{2m+1}\cos^m\left(\frac{4s\pi}{2m+1}-\theta\right)\cos\left(\frac{2s\pi}{2m+1}-\phi\right)\\ &= c_m|\boldsymbol{q}(\boldsymbol{A})|^m\cos(m\theta+\phi), \end{aligned}$$

where c_m is a non-vanishing constant merely depending on m; $\boldsymbol{l}$ is any unit vector in the $\boldsymbol{n}$-plane, and θ and ϕ are the angles between $\boldsymbol{q}(\boldsymbol{A})$ and $\boldsymbol{e}$ and between $\boldsymbol{l}$ and $\boldsymbol{e}$, respectively. Moreover, replacing $\mathring{\boldsymbol{A}}$ with $\boldsymbol{z}\otimes\boldsymbol{z}$ in the above, we gain

$$\boldsymbol{\xi}_m(\boldsymbol{z})\cdot\boldsymbol{l} = d_m|\boldsymbol{z}|^{2m}\cos(2m\theta_0+\phi), \tag{6.9}$$

where θ_0 is the angle between $\boldsymbol{z}$ and $\boldsymbol{e}$. In the above, we mention that $2|\boldsymbol{q}(\boldsymbol{z}\otimes\boldsymbol{z})| = |\boldsymbol{z}|^2$ and the angle between $\boldsymbol{q}(\boldsymbol{z}\otimes\boldsymbol{z})$ and $\boldsymbol{e}$ is given by $2\theta_0$. Since for the vector $\boldsymbol{r}$, $\mathring{\boldsymbol{r}}=\boldsymbol{0}$ is equivalent to $\mathring{\boldsymbol{r}}\cdot\boldsymbol{e}_1=\mathring{\boldsymbol{r}}\cdot\boldsymbol{e}_2=0$, by taking $\boldsymbol{l}=\boldsymbol{e}_1,\boldsymbol{e}_2$, i.e. $\phi=0,\pi/2$, in the last two expressions above we infer that (6.8) is true. Here $\boldsymbol{e}_1=\boldsymbol{e}$ and $\boldsymbol{e}_2$ are two orthonormal vectors in the $\boldsymbol{n}$-plane.

Then, applying (6.4)–(6.5) and (6.8)–(6.9), we deduce

$$\begin{aligned} \gamma_m(\boldsymbol{W}\boldsymbol{n}) = \gamma'_m(\boldsymbol{W}\boldsymbol{n}) = 0 &\Longleftrightarrow \boldsymbol{\xi}_m(\boldsymbol{W}\boldsymbol{n})=\boldsymbol{0} \Longleftrightarrow \boldsymbol{W}\boldsymbol{n}=\boldsymbol{0};\\ \gamma_m(\mathring{\boldsymbol{A}}\boldsymbol{n}) = \gamma'_m(\mathring{\boldsymbol{A}}\boldsymbol{n}) = 0 &\Longleftrightarrow \boldsymbol{\xi}_m(\mathring{\boldsymbol{A}}\boldsymbol{n})=\boldsymbol{0} \Longleftrightarrow \mathring{\boldsymbol{A}}\boldsymbol{n}=\boldsymbol{0}. \end{aligned}$$

Thus, from the above facts and $(6.8)_2$ and

$$g(\boldsymbol{W})\cap C_{2m+1h}(\boldsymbol{n}) = \begin{cases} C_{2m+1h}(\boldsymbol{n}), & \boldsymbol{W}\boldsymbol{n}=\boldsymbol{0},\\ C_1, & \boldsymbol{W}\boldsymbol{n}\neq\boldsymbol{0}, \end{cases} \tag{6.10}$$

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$$g(\boldsymbol{A}) \cap C_{2m+1h}(\boldsymbol{n}) = \begin{cases} C_{2m+1h}(\boldsymbol{n}), & \boldsymbol{A} = x\boldsymbol{I} + y\boldsymbol{n} \otimes \boldsymbol{n}, \\ C_{1h}(\boldsymbol{n}), & \mathring{\boldsymbol{A}}\boldsymbol{n} = \boldsymbol{0}, \quad \boldsymbol{q}(\boldsymbol{A}) \neq \boldsymbol{0}, \\ C_1, & \mathring{\boldsymbol{A}}\boldsymbol{n} \neq \boldsymbol{0}, \end{cases} \tag{6.11}$$

as well as (1.2), we conclude that $V_m^2(\boldsymbol{W})$ and $V_m^2(\boldsymbol{A})$ obey the criterion (1.1) separately.

Furthermore, let $\boldsymbol{W}_i = \boldsymbol{n} \wedge \boldsymbol{e}_i$, $\boldsymbol{A}_i = \boldsymbol{n} \vee \boldsymbol{e}_i$, $i = 1, 2$. Then

$$\gamma'_m(\boldsymbol{W}_1\boldsymbol{n}) = 0, \qquad \gamma_m(\boldsymbol{W}_2\boldsymbol{n}) = 0, \qquad \gamma'_m(\mathring{\boldsymbol{A}}_1\boldsymbol{n}) = 0, \qquad \gamma_m(\mathring{\boldsymbol{A}}_2\boldsymbol{n}) = 0,$$

with $g(\boldsymbol{x}) \cap C_{2m+1h}(\boldsymbol{n}) = C_1$ for $\boldsymbol{x} = \boldsymbol{W}_1, \boldsymbol{W}_2, \boldsymbol{A}_1, \boldsymbol{A}_2$. Thus, we infer that both $V_m^2(\boldsymbol{W})$ and $V_m^2(\boldsymbol{A})$ are irreducible.

Case 3. $\exists \boldsymbol{u}, \boldsymbol{v} \in X : g(X) \cap C_{2m+1h}(\boldsymbol{n}) = g(\boldsymbol{u}, \boldsymbol{v}) \cap C_{2m+1h}(\boldsymbol{n}) \neq g(\boldsymbol{x}) \cap C_{2m+1h}(\boldsymbol{n})$, $\boldsymbol{x} = \boldsymbol{u}, \boldsymbol{v}$.

From (6.1) and the above conditions we derive that (4.16) holds. It can easily be verified that the three irreducible generating sets given in the table for Case 4 in § 4 *a* are also irreducible generating sets for vector-valued, skewsymmetric and symmetric tensor-valued anisotropic functions of the two vector variables $(\boldsymbol{u}, \boldsymbol{v})$ specified by (4.16) under the group $C_{2m+1h}(\boldsymbol{n})$. Thus, the desired result for the case at issue can be obtained from the table for Case 4 in § 4 *a* with $\delta_{1m} = 0$ and with the functional basis placed in the angle brackets by the following functional basis of $(\boldsymbol{u}, \boldsymbol{v})$ under $C_{2m+1h}(\boldsymbol{n})$ (here the first case of (4.16) is considered):

$$\{(\boldsymbol{v} \cdot \boldsymbol{n})^2, \gamma_m(\mathring{\boldsymbol{u}}), \gamma'_m(\mathring{\boldsymbol{u}})\}.$$

Case 4. $\exists (\boldsymbol{u}, \boldsymbol{A}) \in X : g(X) \cap C_{2m+1h}(\boldsymbol{n}) = g(\boldsymbol{u}, \boldsymbol{A}) \cap C_{2m+1h}(\boldsymbol{h}) \neq g(\boldsymbol{x}) \cap C_{2m+1h}(\boldsymbol{n}), \boldsymbol{x} = \boldsymbol{u}, \boldsymbol{A}$.

From (6.1) and (6.12) and the above conditions we derive

$$\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}, \qquad \boldsymbol{u} \neq \boldsymbol{0}, \qquad \mathring{\boldsymbol{A}}\boldsymbol{n} = \boldsymbol{0}, \qquad \mathring{\boldsymbol{A}} \neq \boldsymbol{O},$$

i.e.

$$\left.\begin{aligned} &\boldsymbol{u} = a\boldsymbol{n}, \qquad a \neq 0, \\ &\boldsymbol{A} = x(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_2) + y\boldsymbol{e}_1 \vee \boldsymbol{e}_2 + c\boldsymbol{I} + d\boldsymbol{n} \otimes \boldsymbol{n}, x^2 + y^2 \neq 0. \end{aligned}\right\} \tag{6.12}$$

It may readily be verified that for the variables $(\boldsymbol{u}, \boldsymbol{A})$ specified above, the union $\{(\boldsymbol{u}\cdot\boldsymbol{n})^2\} \cup I_m^2(\boldsymbol{A})$ supplies an irreducible functional basis of $(\boldsymbol{u}, \boldsymbol{A})$ under $C_{2m+1h}(\boldsymbol{n})$. On the other hand, for the variables $(\boldsymbol{u}, \boldsymbol{A})$ at issue, generating sets for vector-valued and second-order tensor-valued anisotropic functions of the variables $(\boldsymbol{u}, \boldsymbol{A})$ under $C_{2m+1h}(\boldsymbol{n})$ can be derived from vector-valued and second-order tensor-valued isotropic functions of the extended variables

$$(\boldsymbol{u}, \boldsymbol{A}, \boldsymbol{\eta}_m(\mathring{\boldsymbol{A}}), \boldsymbol{N})$$

(see Xiao 1995*a*, 1996*a*). Thus, we construct the following table for irreducible generating sets and the related invariants:

V	$(\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n}, \boldsymbol{\eta}_m(\mathring{\boldsymbol{A}}), \boldsymbol{n} \times \boldsymbol{\eta}_m(\mathring{\boldsymbol{A}}),$
Skw	$\boldsymbol{N}, (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n} \wedge \boldsymbol{\eta}_m(\mathring{\boldsymbol{A}}), (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n} \wedge (\boldsymbol{n} \times \boldsymbol{\eta}_m(\mathring{\boldsymbol{A}})),$

Sym	$\boldsymbol{I}, \boldsymbol{n}\otimes\boldsymbol{n}, \boldsymbol{A}, \boldsymbol{AN}-\boldsymbol{NA}, (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}\vee\boldsymbol{\eta}_m(\mathring{\boldsymbol{A}}), (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}\vee(\boldsymbol{n}\times\boldsymbol{\eta}_m(\mathring{\boldsymbol{A}}))$,
R	$(\boldsymbol{u}\cdot\boldsymbol{n})(\boldsymbol{r}\cdot\boldsymbol{n}); \operatorname{tr}\boldsymbol{HN}; \operatorname{tr}\boldsymbol{C}, \boldsymbol{n}\cdot\boldsymbol{Cn}, \operatorname{tr}\boldsymbol{AC}, \operatorname{tr}\boldsymbol{ACN}$
	$\langle(\boldsymbol{u}\cdot\boldsymbol{n})^2, I_m^2(\boldsymbol{A})\rangle$.

The proof for the above result is easy. In the above table, we regard the generic variables $\boldsymbol{r}\in V$, $\boldsymbol{H}\in$ Skw and $\boldsymbol{C}\in$ Sym as being subjected to the conditions $\boldsymbol{r}\times\boldsymbol{n}=\boldsymbol{Hn}=\mathring{\boldsymbol{C}}\boldsymbol{n}=\boldsymbol{0}$, for the other cases have been covered before. As a result, when forming the inner product between each presented generator and each of these variables, only those listed in the above table are non-vanishing.

Finally, combining the above cases, we arrive at the main result of this section as follows.

Theorem 6.1. *The four sets given by*

$I_m^2(\boldsymbol{u}); I_{2m+1}^0(\boldsymbol{W}); I_{2m+1}^0(\boldsymbol{A})$;
$\boldsymbol{u}\cdot\boldsymbol{\xi}_m(\boldsymbol{Wn}), [\boldsymbol{n},\boldsymbol{u},\boldsymbol{\xi}_m(\boldsymbol{Wn})], (\boldsymbol{u}\cdot\boldsymbol{n})\gamma_m(\boldsymbol{Wn}), (\boldsymbol{u}\cdot\boldsymbol{n})\gamma'_m(\boldsymbol{Wn})$;
$\boldsymbol{u}\cdot\boldsymbol{\xi}_m(\mathring{\boldsymbol{A}}\boldsymbol{n}), [\boldsymbol{n},\boldsymbol{u},\boldsymbol{\xi}_m(\mathring{\boldsymbol{A}}\boldsymbol{n})], (\boldsymbol{u}\cdot\boldsymbol{n})\gamma_m(\mathring{\boldsymbol{A}}\boldsymbol{n}), (\boldsymbol{u}\cdot\boldsymbol{n})\gamma'_m(\mathring{\boldsymbol{A}}\boldsymbol{n})$;
$(\boldsymbol{u}\cdot\boldsymbol{n})(\boldsymbol{v}\cdot\boldsymbol{n}), \mathring{\boldsymbol{u}}\cdot\mathring{\boldsymbol{v}}, [\boldsymbol{n},\mathring{\boldsymbol{u}},\mathring{\boldsymbol{v}}]; (\boldsymbol{u}\cdot\boldsymbol{n})\mathring{\boldsymbol{u}}\cdot\boldsymbol{Wn}, (\boldsymbol{u}\cdot\boldsymbol{n})[\boldsymbol{n},\boldsymbol{Wn},\mathring{\boldsymbol{u}}]$;
$(\boldsymbol{u}\cdot\boldsymbol{n})\mathring{\boldsymbol{u}}\cdot\boldsymbol{An}, (\boldsymbol{u}\cdot\boldsymbol{n})[\boldsymbol{n},\boldsymbol{An},\mathring{\boldsymbol{u}}], \mathring{\boldsymbol{u}}\cdot\boldsymbol{A}\mathring{\boldsymbol{u}}, [\boldsymbol{n},\mathring{\boldsymbol{u}},\boldsymbol{A}\mathring{\boldsymbol{u}}]$;
$\boldsymbol{n}\cdot\boldsymbol{WHn}, [\boldsymbol{n},\boldsymbol{Wn},\boldsymbol{Hn}]; \boldsymbol{n}\cdot\boldsymbol{WAn}, [\boldsymbol{n},\boldsymbol{Wn},\boldsymbol{An}], \boldsymbol{n}\cdot\boldsymbol{WAWn}, [\boldsymbol{n},\boldsymbol{Wn},\boldsymbol{AWn}]$;
$\operatorname{tr}\boldsymbol{AB}, \operatorname{tr}\boldsymbol{ABN}, \boldsymbol{n}\cdot\mathring{\boldsymbol{A}}\boldsymbol{B}\mathring{\boldsymbol{A}}\boldsymbol{n}, [\boldsymbol{n},\mathring{\boldsymbol{A}}\boldsymbol{n},\boldsymbol{B}\mathring{\boldsymbol{A}}\boldsymbol{n}]$;
$\boldsymbol{u}\cdot\boldsymbol{Wv}, [\boldsymbol{n},\boldsymbol{u},\boldsymbol{Wv}]+[\boldsymbol{n},\boldsymbol{v},\boldsymbol{Wu}]; \boldsymbol{u}\cdot\boldsymbol{Av}, [\boldsymbol{n},\boldsymbol{u},\boldsymbol{Av}]+[\boldsymbol{n},\boldsymbol{v},\boldsymbol{Au}]$;

and

$$V^0(\boldsymbol{u}); V_m^2(\boldsymbol{W}); V_m^2(\boldsymbol{A});$$

and

$$\mathrm{Skw}^0(\boldsymbol{u}); \mathrm{Skw}^0(\boldsymbol{W}); \mathrm{Skw}^0(\boldsymbol{A});$$
$$\boldsymbol{u}\wedge\boldsymbol{v}, (\boldsymbol{u}\times\boldsymbol{n})\wedge\boldsymbol{v}+(\boldsymbol{v}\times\boldsymbol{n})\wedge\boldsymbol{u}; (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{\eta}_m(\mathring{\boldsymbol{A}}), (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}\wedge(\boldsymbol{n}\times\boldsymbol{\eta}_m(\mathring{\boldsymbol{A}}));$$

and

$$\mathrm{Sym}_{2m+1}^0(\boldsymbol{u}); \mathrm{Sym}_{2m+1}^0(\boldsymbol{W}); \mathrm{Sym}_{2m+1}^0(\boldsymbol{A});$$
$$\boldsymbol{u}\vee\boldsymbol{v}, (\boldsymbol{u}\times\boldsymbol{n})\vee\boldsymbol{v}+(\boldsymbol{v}\times\boldsymbol{n})\vee\boldsymbol{u}; (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{\eta}_m(\mathring{\boldsymbol{A}}), (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}\vee(\boldsymbol{n}\times\boldsymbol{\eta}_m(\mathring{\boldsymbol{A}}));$$

where $\boldsymbol{u},\boldsymbol{v}=\boldsymbol{u}_1,\ldots,\boldsymbol{u}_a$; $\boldsymbol{W},\boldsymbol{H}=\boldsymbol{W}_1,\ldots,\boldsymbol{W}_b$; $\boldsymbol{A},\boldsymbol{B}=\boldsymbol{A}_1,\ldots,\boldsymbol{A}_c$; $\boldsymbol{u}\neq\boldsymbol{v}$, $\boldsymbol{W}\neq\boldsymbol{H}$, *provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables* $X\in\mathcal{D}$ *under the group* $C_{2m+1h}(\boldsymbol{n})$ *for each integer* $m\geqslant 1$, *respectively.*

7. The classes S_{4m+2} and C_{2m+1}

The classes at issue include the trigonal crystal classes S_6 and C_3 in the particular cases when $m=1$.

(*a*) *The classes* S_{4m+2}

Applying the facts

$$g(\boldsymbol{r})\cap S_{4m+2}(\boldsymbol{n})=\begin{cases}S_{4m+2}(\boldsymbol{n}), & \boldsymbol{r}=\boldsymbol{0},\\ C_{2m+1}(\boldsymbol{n}), & \boldsymbol{r}=a\boldsymbol{n}\neq\boldsymbol{0},\\ C_1, & \boldsymbol{r}\times\boldsymbol{n}\neq\boldsymbol{0},\end{cases} \tag{7.1}$$

for any vector $\boldsymbol{r} \in V$, we infer

$$g(\boldsymbol{u}, \boldsymbol{v}) \cap S_{4m+2}(\boldsymbol{n}) = g(\boldsymbol{z}) \cap S_{4m+2}(\boldsymbol{n}), \boldsymbol{z} \in \{\boldsymbol{u}, \boldsymbol{v}\}$$

for any two vectors $\boldsymbol{u}, \boldsymbol{v} \in V$. Thus, it suffices to treat the cases for a single variable and two variables: $\boldsymbol{u}$, $\boldsymbol{W}$, $\boldsymbol{A}$, $(\boldsymbol{u}, \boldsymbol{W})$ and $(\boldsymbol{u}, \boldsymbol{A})$.

Case 1. $\exists \boldsymbol{u} \in X : g(X) \cap S_{4m+2}(\boldsymbol{n}) = g(\boldsymbol{u}) \cap S_{4m+2}(\boldsymbol{n})$.

A scalar-valued (resp. second-order tensor-valued) anisotropic function of the vector variable $\boldsymbol{u}$ under $S_{4m+2}(\boldsymbol{n})$ is equivalent to a scalar-valued (resp. second-order tensor-valued) anisotropic function of the symmetric tensor variable $\boldsymbol{u} \otimes \boldsymbol{u}$ under $S_{4m+2}(\boldsymbol{n})$. As a result, generating sets for the former can be derived by setting $\boldsymbol{A} = \boldsymbol{u} \otimes \boldsymbol{u}$ in the corresponding generating sets listed in the table for Case 3 given later. Moreover, generating sets for vector-valued anisotropic functions of $\boldsymbol{u}$ under $S_{4m+2}(\boldsymbol{n})$ can be obtained from those for vector-valued isotropic functions of the extended variables $(\boldsymbol{u}, \boldsymbol{E}\boldsymbol{\xi}_m(\mathring{\boldsymbol{u}}), \boldsymbol{N})$, where $\boldsymbol{\xi}_m(\mathring{\boldsymbol{u}})$ is given by $(6.6)_1$ with $\boldsymbol{x} = \mathring{\boldsymbol{u}}$. Applying these facts, we construct the following table:

V	$\boldsymbol{u}, \mathring{\boldsymbol{u}} \times \boldsymbol{n}, \gamma_m(\mathring{\boldsymbol{u}})\boldsymbol{n}, \gamma'_m(\mathring{\boldsymbol{u}})\boldsymbol{n},$
Skw	$\boldsymbol{N}, \boldsymbol{n} \wedge \boldsymbol{\xi}_m(\mathring{\boldsymbol{u}}), \boldsymbol{n} \wedge (\boldsymbol{n} \times \boldsymbol{\xi}_m(\mathring{\boldsymbol{u}})),$
Sym	$\boldsymbol{I}, \boldsymbol{n} \otimes \boldsymbol{n}, \mathring{\boldsymbol{u}} \otimes \mathring{\boldsymbol{u}}, \mathring{\boldsymbol{u}} \vee (\boldsymbol{n} \times \mathring{\boldsymbol{u}}), \boldsymbol{n} \vee \boldsymbol{\xi}_m(\mathring{\boldsymbol{u}}), \boldsymbol{n} \vee (\boldsymbol{n} \times \boldsymbol{\xi}_m(\mathring{\boldsymbol{u}})),$
R	$\boldsymbol{u} \cdot \boldsymbol{r}, [\boldsymbol{n}, \mathring{\boldsymbol{u}}, \mathring{\boldsymbol{r}}], (\boldsymbol{r} \cdot \boldsymbol{n})\gamma_m(\mathring{\boldsymbol{u}}), (\boldsymbol{r} \cdot \boldsymbol{n})\gamma'_m(\mathring{\boldsymbol{u}});$
	$\operatorname{tr} \boldsymbol{H}\boldsymbol{N}, \boldsymbol{n} \cdot \boldsymbol{H}\boldsymbol{\xi}_m(\mathring{\boldsymbol{u}}), [\boldsymbol{n}, \boldsymbol{H}\boldsymbol{n}, \boldsymbol{\xi}_m(\mathring{\boldsymbol{u}})]$
	$\operatorname{tr} \boldsymbol{C}, \boldsymbol{n} \cdot \boldsymbol{C}\boldsymbol{n}, \mathring{\boldsymbol{u}} \cdot \boldsymbol{C}\mathring{\boldsymbol{u}}, [\boldsymbol{n}, \mathring{\boldsymbol{u}}, \boldsymbol{C}\mathring{\boldsymbol{u}}], \boldsymbol{n} \cdot \boldsymbol{C}\boldsymbol{\xi}_m(\mathring{\boldsymbol{u}}), [\boldsymbol{n}, \boldsymbol{C}\boldsymbol{n}, \boldsymbol{\xi}_m(\mathring{\boldsymbol{u}})]$
	$\langle (\boldsymbol{u} \cdot \boldsymbol{n})^2, (\boldsymbol{u} \cdot \boldsymbol{n})\gamma_m(\mathring{\boldsymbol{u}}), (\boldsymbol{u} \cdot \boldsymbol{n})\gamma'_m(\mathring{\boldsymbol{u}}), \alpha_{2m+1}(\mathring{\boldsymbol{u}}), \alpha'_{2m+1}(\mathring{\boldsymbol{u}}) \rangle.$

With the aid of (7.1) and (6.11) with $C_{2m+1h}(\boldsymbol{n})$ replaced by $S_{4m+2}(\boldsymbol{n})$, as well as (1.2)–(1.4), it can be proved that the first three sets in the above table, denoted by $V^3_m(\boldsymbol{u})$, $\mathrm{Skw}^3_m(\boldsymbol{u})$ and $\mathrm{Sym}^3_m(\boldsymbol{u})$ henceforth, are irreducible generating sets required. Moreover, the functional basis $I^3_m(\boldsymbol{u})$, given in the angle brackets, is derived by setting $\boldsymbol{A} = \boldsymbol{u} \otimes \boldsymbol{u}$ in the functional basis of $\boldsymbol{A} \in \mathrm{Sym}$ given in the table for Case 3 below, as stated before.

Case 2. $\exists \boldsymbol{W} \in X : g(X) \cap S_{4m+2}(\boldsymbol{n}) = g(\boldsymbol{W}) \cap S_{4m+2}(\boldsymbol{n})$.

Every vector-valued anisotropic function of $\boldsymbol{W}$ vanishes. Anisotropic functional bases of the variable $\boldsymbol{W} \in \mathrm{Skw}$ under $S_{4m+2}(\boldsymbol{n})$ can be obtained from isotropic functional bases of the extended variables $(\boldsymbol{W}, \boldsymbol{n} \wedge \boldsymbol{\xi}_m(\boldsymbol{W}\boldsymbol{n}), \boldsymbol{N})$ (see Xiao 1995*a*, 1996*a*). Moreover, it can be shown that the sets $\mathrm{Skw}^0(\boldsymbol{W})$ and $\mathrm{Sym}^0_{2m+1}(\boldsymbol{W})$ (see Case 2 in § 4 *a*) provide irreducible generating sets for skewsymmetric and symmetric tensor-valued anisotropic functions of the variable $\boldsymbol{W}$ under $S_{4m+2}(\boldsymbol{n})$. Thus, here we only need to supply an irreducible functional basis of $\boldsymbol{W}$ under $S_{4m+2}(\boldsymbol{n})$. Other relevant invariants can be found in the table for Case 2 in § 4 *a*.

$$I^3_m(\boldsymbol{W}) = \{\operatorname{tr} \boldsymbol{W}\boldsymbol{N}, \gamma_m(\boldsymbol{W}\boldsymbol{n}), \gamma'_m(\boldsymbol{W}\boldsymbol{n})\}. \tag{7.2}$$

By means of the identity (4.6) and the procedure used at Case 1 in § 6 *a*, it can be proved that the above set is indeed a desired irreducible functional basis.

Case 3. $\forall \boldsymbol{H} \in X : \boldsymbol{H}\boldsymbol{n} = \boldsymbol{0} \& \exists \boldsymbol{A} \in X : g(X) \cap S_{4m+2}(\boldsymbol{n}) = g(\boldsymbol{A}) \cap S_{4m+2}(\boldsymbol{n})$.

Every vector-valued anisotropic function of $\boldsymbol{W}$ vanishes. Irreducible representations for scalar-valued and second-order tensor-valued anisotropic functions of the variable $\boldsymbol{A} \in \mathrm{Sym}$ under $S_{4m+2}(\boldsymbol{n})$ are obtainable from those for scalar-valued and second-order tensor-valued isotropic functions of the extended variables

$$(\boldsymbol{A}, \boldsymbol{E}\boldsymbol{\xi}_m(\mathring{\boldsymbol{A}}\boldsymbol{n}), \boldsymbol{E}\boldsymbol{\eta}_m(\mathring{\boldsymbol{A}}), \boldsymbol{N})$$

(see Xiao 1995*a*, 1996*a*), where $\boldsymbol{\xi}_m(\mathring{\boldsymbol{A}}\boldsymbol{n})$ and $\boldsymbol{\eta}_m(\mathring{\boldsymbol{A}})$ are given by (6.6)–(6.7). Taking these facts into account, we construct the following table:

$$\begin{array}{ll}
\mathrm{Skw} & \boldsymbol{N}, \boldsymbol{n} \wedge \mathring{\boldsymbol{A}}\boldsymbol{n}, \boldsymbol{n} \wedge (\boldsymbol{n} \times \mathring{\boldsymbol{A}}\boldsymbol{n}), \boldsymbol{n} \wedge \boldsymbol{\eta}_m(\mathring{\boldsymbol{A}}), \boldsymbol{n} \wedge (\boldsymbol{n} \times \boldsymbol{\eta}_m(\mathring{\boldsymbol{A}})), \\
\mathrm{Sym} & \boldsymbol{I}, \boldsymbol{n} \otimes \boldsymbol{n}, \boldsymbol{A}, \boldsymbol{AN} - \boldsymbol{NA}, \boldsymbol{n} \vee \mathring{\boldsymbol{A}}\boldsymbol{n}, \boldsymbol{n} \vee (\boldsymbol{n} \times \mathring{\boldsymbol{A}}\boldsymbol{n}), \\
 & \boldsymbol{n} \vee \boldsymbol{\eta}_m(\mathring{\boldsymbol{A}}), \boldsymbol{n} \vee (\boldsymbol{n} \times \boldsymbol{\eta}_m(\mathring{\boldsymbol{A}})), \\
R & \mathrm{tr}\,\boldsymbol{HN}; \mathrm{tr}\,\boldsymbol{C}, \boldsymbol{n} \cdot \boldsymbol{Cn}, \mathrm{tr}\,\boldsymbol{AC}, \mathrm{tr}\,\boldsymbol{ACN}, \\
 & (\mathring{\boldsymbol{A}}\boldsymbol{n}) \cdot (\mathring{\boldsymbol{C}}\boldsymbol{n}), [\boldsymbol{n}, \mathring{\boldsymbol{A}}\boldsymbol{n}, \mathring{\boldsymbol{C}}\boldsymbol{n}], \boldsymbol{n} \cdot \boldsymbol{C}\boldsymbol{\eta}_m(\mathring{\boldsymbol{A}}), [\boldsymbol{n}, \mathring{\boldsymbol{C}}\boldsymbol{n}, \boldsymbol{\eta}_m(\mathring{\boldsymbol{A}})] \\
 & \langle \mathrm{tr}\,\boldsymbol{A}, \mathrm{tr}\,\boldsymbol{A}^3, \boldsymbol{n} \cdot \boldsymbol{n}, [\boldsymbol{n}, \boldsymbol{An}, \boldsymbol{A}^2\boldsymbol{n}], \gamma_m(\mathring{\boldsymbol{A}}\boldsymbol{n}), \gamma'_m(\mathring{\boldsymbol{A}}\boldsymbol{n}), \beta_{2m+1}(\mathring{\boldsymbol{A}}), \beta'_{2m+1}(\mathring{\boldsymbol{A}}) \rangle.
\end{array}$$

By means of the criterion (1.1) and (1.3)–(1.4), as well as

$$g(\boldsymbol{A}) \cap S_{4m+2}(\boldsymbol{n}) = \begin{cases} S_{4m+2}(\boldsymbol{n}), & \mathring{\boldsymbol{A}} = \boldsymbol{O}, \\ C_1, & \mathring{\boldsymbol{A}} \neq \boldsymbol{O}, \end{cases}$$

it can be proved that the first two sets of generators in the above table, denoted by $\mathrm{Skw}^3_m(\boldsymbol{A})$ and $\mathrm{Sym}^3_m(\boldsymbol{A})$ henceforth, are generating sets for skewsymmetric and symmetric tensor-valued anisotropic functions of the variable $\boldsymbol{A} \in \mathrm{Sym}$ under $S_{4m+2}(\boldsymbol{n})$, and that the two generating sets are irreducible. In what follows, we show that the set placed in the angle brackets, denoted by $I^3_m(\boldsymbol{A})$ henceforth, is an irreducible functional basis of $\boldsymbol{A}$ under $S_{4m+2}(\boldsymbol{n})$.

In fact, by following the same procedure as that used for deriving the similar result at Case 3 in § 4 *a*, we infer the following fact holds: for $\bar{\boldsymbol{A}}, \boldsymbol{A} \in \mathrm{Sym}$,

$$I^3_m(\bar{\boldsymbol{A}}) = I^3_m(\boldsymbol{A}) \Longrightarrow I_\infty(\bar{\boldsymbol{A}}) = I_\infty(\boldsymbol{A}) \Longrightarrow \exists \boldsymbol{Q} = \boldsymbol{R}^\psi_{\boldsymbol{n}} \in C_{\infty h}(\boldsymbol{n}) : \bar{\boldsymbol{A}} = \boldsymbol{QAQ}^{\mathrm{T}}.$$

Suppose $\mathring{\boldsymbol{A}}\boldsymbol{n} \neq \boldsymbol{0}$. Then, using the same procedure as that used at Case 1 in § 6, we derive

$$\bar{\theta} = \frac{2k\pi}{2m+1} + \theta, \qquad k = 0, \pm 1, \pm 2, \ldots,$$

where $\bar{\theta}$ and θ are the angles between $\mathring{\bar{\boldsymbol{A}}}\boldsymbol{n}$ and $\boldsymbol{e}$ and between $\mathring{\boldsymbol{A}}\boldsymbol{n}$ and $\boldsymbol{e}$, respectively. From the equality

$$\mathring{\bar{\boldsymbol{A}}}\boldsymbol{n} = \boldsymbol{R}^\psi_{\boldsymbol{n}} \mathring{\boldsymbol{A}} \boldsymbol{R}^{-\psi}_{\boldsymbol{n}} \boldsymbol{n} = \boldsymbol{R}^\psi_{\boldsymbol{n}}(\mathring{\boldsymbol{A}}\boldsymbol{n}),$$

we get $\bar{\theta} = \theta + \psi$. Thus, $\psi = 2k\pi/(2m+1)$, i.e.

$$I^3_m(\bar{\boldsymbol{A}}) = I^3_m(\boldsymbol{A}) \Longrightarrow \exists \boldsymbol{Q} \in S_{4m+2}(\boldsymbol{n}) : \bar{\boldsymbol{A}} = \boldsymbol{QAQ}^{\mathrm{T}}. \tag{7.3}$$

Suppose $\mathring{\boldsymbol{A}}\boldsymbol{n} = \boldsymbol{0}$. Then for any scalar-valued function $f(\boldsymbol{A})$ we have

$$f(\boldsymbol{QAQ}^{\mathrm{T}}) = f(\boldsymbol{A}), \boldsymbol{Q} = \boldsymbol{R}^\pi_{\boldsymbol{n}}.$$

The above fact implies that for the variable $\boldsymbol{A}$ with $\mathring{\boldsymbol{A}}\boldsymbol{n} = \boldsymbol{0}$, each invariant of the variable $\boldsymbol{A}$ under the group $S_{4m+2}(\boldsymbol{n})$ turns out to be an invariant of $\boldsymbol{A}$ under the larger group $C_{4m+2h}(\boldsymbol{n})$ ($\supset S_{4m+2}(\boldsymbol{n})$). Consequently, for the variable $\boldsymbol{A}$ with $\mathring{\boldsymbol{A}}\boldsymbol{n} = \boldsymbol{0}$, a functional basis of $\boldsymbol{A}$ under $S_{4m+2}(\boldsymbol{n})$ may be provided by $I^0_{2m+1}(\boldsymbol{A})$ (see the table for Case 3 in § 4 *a*). Here the two invariants $\alpha_{2m+1}(\mathring{\boldsymbol{A}}\boldsymbol{n})$ and $\alpha'_{2m+1}(\mathring{\boldsymbol{A}}\boldsymbol{n})$ vanish and hence only the two invariants $\beta_{2m+1}(\mathring{\boldsymbol{A}})$ and $\beta_{2m+1}(\mathring{\boldsymbol{A}})$ are retained. From the above, we conclude that $I^3_m(\boldsymbol{A})$ is a functional basis of the variable $\boldsymbol{A} \in \mathrm{Sym}$ under $S_{4m+2}(\boldsymbol{n})$. Moreover, again from the eight pairs $X = \boldsymbol{A}$ and $X' = \boldsymbol{A}'$ given at the end of Case 3 in § 4 *a*, where m is replaced by $2m+1$, one can infer that the basis $I^3_m(\boldsymbol{A})$ is irreducible.

Case 4. $\exists(\boldsymbol{u}, \boldsymbol{W}) \subset X : g(X) \cap S_{4m+2}(\boldsymbol{n}) = g(\boldsymbol{u}, \boldsymbol{W}) \cap S_{4m+2}(\boldsymbol{n}) \neq g(\boldsymbol{x}) \cap S_{4m+2}(\boldsymbol{n}), \boldsymbol{x} = \boldsymbol{u}, \boldsymbol{W}$.

From (7.1) and the above conditions we derive

$$\left.\begin{aligned} \boldsymbol{u} = a\boldsymbol{n} \neq \boldsymbol{0}, \\ \boldsymbol{W}\boldsymbol{n} \neq \boldsymbol{0},\ \text{i.e. } g(\boldsymbol{W}) \cap S_{4m+2}(\boldsymbol{n}) = S_2. \end{aligned}\right\} \tag{7.4}$$

As a result, the sets $\mathrm{Skw}^0(\boldsymbol{W})$ and $\mathrm{Sym}^0_{2m+1}(\boldsymbol{W})$ supply desired irreducible generating sets. Moreover, it is evident that the union $I^3_m(\boldsymbol{W}) \cup \{(\boldsymbol{u}\cdot\boldsymbol{n})^2\}$ provides an irreducible functional basis of the variables $(\boldsymbol{u}, \boldsymbol{W})$ specified by (7.4) under $S_{4m+2}(\boldsymbol{n})$. Thus, here we need only to provide an irreducible generating set for vector-valued functions of the variables $(\boldsymbol{u}, \boldsymbol{W})$ with (7.4) under $S_{4m+2}(\boldsymbol{n})$, which is as follows.

$$V^3_m(\boldsymbol{W}) = \{\boldsymbol{u}, (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{\eta}_m(\boldsymbol{W}\boldsymbol{n}), (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{\eta}_m(\boldsymbol{W}\boldsymbol{n})\}. \tag{7.5}$$

Case 5. $\exists(\boldsymbol{u}, \boldsymbol{A}) \subset X : g(X) \cap S_{4m+2}(\boldsymbol{n}) = g(\boldsymbol{u}, \boldsymbol{A}) \cap S_{4m+2}(\boldsymbol{n}) \neq g(\boldsymbol{x}) \cap S_{4m+2}(\boldsymbol{n}), \boldsymbol{x} = \boldsymbol{u}, \boldsymbol{A}$.

From (7.1) and the above conditions we derive

$$\left.\begin{aligned} \boldsymbol{u} = a\boldsymbol{n} \neq \boldsymbol{0}, \\ \mathring{\boldsymbol{A}}\boldsymbol{n} \neq \boldsymbol{0},\ \text{i.e. } g(\boldsymbol{A}) \cap S_{4m+2}(\boldsymbol{n}) = S_2. \end{aligned}\right\} \tag{7.6}$$

As a result, the sets $\mathrm{Skw}^0(\boldsymbol{A})$ and $\mathrm{Sym}^0_{2m+1}(\boldsymbol{A})$ supply desired irreducible generating sets. Moreover, it is evident that the union $I^3_m(\boldsymbol{A}) \cup \{(\boldsymbol{u}\cdot\boldsymbol{n})^2\}$ provides an irreducible functional basis of the variables $(\boldsymbol{u}, \boldsymbol{A})$ specified by (7.6) under $S_{4m+2}(\boldsymbol{n})$. Thus, here we need only to provide an irreducible generating set for vector-valued functions of the variables $(\boldsymbol{u}, \boldsymbol{A})$ with (7.6) under $S_{4m+2}(\boldsymbol{n})$, which is as follows.

$$V^3_m(\boldsymbol{A}) = \{\boldsymbol{u}, (\boldsymbol{u}\cdot\boldsymbol{n})\mathring{\boldsymbol{A}}\boldsymbol{n}, (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}\times\mathring{\boldsymbol{A}}\boldsymbol{n}, (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{\eta}_m(\mathring{\boldsymbol{A}}), (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}\times\boldsymbol{\eta}_m(\mathring{\boldsymbol{A}})\}. \tag{7.7}$$

Finally, combining the above cases, we arrive at the main result of this subsection as follows.

Theorem 7.1. *The four sets given by*

$$\begin{aligned}
&I^3_m(\boldsymbol{u}); I^3_m(\boldsymbol{W}); I^3_m(\boldsymbol{A}); \boldsymbol{u}\cdot\boldsymbol{v}, [\boldsymbol{n}, \mathring{\boldsymbol{u}}, \mathring{\boldsymbol{v}}], (\boldsymbol{u}\cdot\boldsymbol{n})\gamma_m(\mathring{\boldsymbol{v}}), (\boldsymbol{u}\cdot\boldsymbol{n})\gamma'_m(\mathring{\boldsymbol{v}});\\
&\boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{H}\boldsymbol{n}, [\boldsymbol{n}, \boldsymbol{W}\boldsymbol{n}, \boldsymbol{H}\boldsymbol{n}]; \boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{A}\boldsymbol{n}, [\boldsymbol{n}, \boldsymbol{W}\boldsymbol{n}, \boldsymbol{A}\boldsymbol{n}], \boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{A}\boldsymbol{W}\boldsymbol{n}, [\boldsymbol{n}, \boldsymbol{W}\boldsymbol{n}, \boldsymbol{A}\boldsymbol{W}\boldsymbol{n}];\\
&\mathrm{tr}\,\boldsymbol{A}\boldsymbol{B}, \mathrm{tr}\,\boldsymbol{A}\boldsymbol{B}\boldsymbol{N}, (\mathring{\boldsymbol{A}}\boldsymbol{n})\cdot(\mathring{\boldsymbol{B}}\boldsymbol{n}), [\boldsymbol{n}, \mathring{\boldsymbol{A}}\boldsymbol{n}, \mathring{\boldsymbol{B}}\boldsymbol{n}], (\mathring{\boldsymbol{A}}\boldsymbol{n})\cdot\boldsymbol{\eta}_m(\mathring{\boldsymbol{B}}), [\boldsymbol{n}, \mathring{\boldsymbol{A}}\boldsymbol{n}), \boldsymbol{\eta}_m(\mathring{\boldsymbol{B}})];\\
&\boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{\xi}_m(\mathring{\boldsymbol{u}}), [\boldsymbol{n}, \boldsymbol{W}\boldsymbol{n}, \boldsymbol{\xi}_m(\mathring{\boldsymbol{u}})]; \mathring{\boldsymbol{u}}\cdot\boldsymbol{A}\mathring{\boldsymbol{u}}, [\boldsymbol{n}, \mathring{\boldsymbol{u}}, \boldsymbol{A}\mathring{\boldsymbol{u}}], (\mathring{\boldsymbol{A}}\boldsymbol{n})\cdot\boldsymbol{\xi}_m(\mathring{\boldsymbol{u}}), [\boldsymbol{n}, \mathring{\boldsymbol{A}}\boldsymbol{n}, \boldsymbol{\xi}_m(\mathring{\boldsymbol{u}})];
\end{aligned}$$

and

$$V^3_m(\boldsymbol{u}); V^3_m(\boldsymbol{u}, \boldsymbol{W}); V^3_m(\boldsymbol{u}, \boldsymbol{A});$$

and

$$\mathrm{Skw}^3_m(\boldsymbol{u}); \mathrm{Skw}^0(\boldsymbol{W}); \mathrm{Skw}^3_m(\boldsymbol{A});$$

and

$$\mathrm{Sym}^3_m(\boldsymbol{u}); \mathrm{Sym}^0_{2m+1}(\boldsymbol{W}); \mathrm{Sym}^3_m(\boldsymbol{A});$$

where $\boldsymbol{u}, \boldsymbol{v} = \boldsymbol{u}_1, \dots, \boldsymbol{u}_a$; $\boldsymbol{W}, \boldsymbol{H} = \boldsymbol{W}_1, \dots, \boldsymbol{W}_b$; $\boldsymbol{A}, \boldsymbol{B} = \boldsymbol{A}_1, \dots, \boldsymbol{A}_c$; $\boldsymbol{u} \neq \boldsymbol{v}$, $\boldsymbol{W} \neq \boldsymbol{H}$, *provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables* $X \in \mathcal{D}$ *under the group* $S_{4m+2}(\boldsymbol{n})$ *for each integer* $m \geqslant 1$, *respectively.*

(*b*) *The classes* C_{2m+1}

Applying the arguments given at the start of § 4 *b* and theorem 7.1, we derive the following result.

Theorem 7.2. *The four sets given by*

$$\boldsymbol{u}\cdot\boldsymbol{n}, \gamma_m(\mathring{\boldsymbol{u}}), \gamma'_m(\mathring{\boldsymbol{u}}); I^3_m(\boldsymbol{W}); I^3_m(\boldsymbol{A}); \boldsymbol{u}\cdot\boldsymbol{v}, [\boldsymbol{n},\boldsymbol{u},\boldsymbol{v}], \boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{H}\boldsymbol{n}, [\boldsymbol{n},\boldsymbol{W}\boldsymbol{n},\boldsymbol{H}\boldsymbol{n}];$$
$$\boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{A}\boldsymbol{n}, [\boldsymbol{n},\boldsymbol{W}\boldsymbol{n},\boldsymbol{A}\boldsymbol{n}], \boldsymbol{n}\cdot\boldsymbol{W}\boldsymbol{A}\boldsymbol{W}\boldsymbol{n}, [\boldsymbol{n},\boldsymbol{W}\boldsymbol{n},\boldsymbol{A}\boldsymbol{W}\boldsymbol{n}];$$
$$\mathrm{tr}\,\boldsymbol{A}\boldsymbol{B}, \mathrm{tr}\,\boldsymbol{A}\boldsymbol{B}\boldsymbol{N}, (\mathring{\boldsymbol{A}}\boldsymbol{n})\cdot(\mathring{\boldsymbol{B}}\boldsymbol{n}), [\boldsymbol{n},\mathring{\boldsymbol{A}}\boldsymbol{n},\mathring{\boldsymbol{B}}\boldsymbol{n}], (\mathring{\boldsymbol{A}}\boldsymbol{n})\cdot\boldsymbol{\eta}_m(\mathring{\boldsymbol{B}}), [\boldsymbol{n},\mathring{\boldsymbol{A}}\boldsymbol{n},\boldsymbol{\eta}_m(\mathring{\boldsymbol{B}})];$$
$$\boldsymbol{u}\cdot\boldsymbol{W}\boldsymbol{n}, [\boldsymbol{n},\boldsymbol{u},\boldsymbol{W}\boldsymbol{n}]; \boldsymbol{u}\cdot\boldsymbol{A}\boldsymbol{n}, [\boldsymbol{n},\boldsymbol{u},\boldsymbol{A}\boldsymbol{n}], \mathring{\boldsymbol{u}}\cdot\boldsymbol{A}\mathring{\boldsymbol{u}}, [\boldsymbol{n},\boldsymbol{u},\boldsymbol{A}\boldsymbol{u}];$$

and

$$\boldsymbol{n}, \mathring{\boldsymbol{u}}, \boldsymbol{n}\times\mathring{\boldsymbol{u}}, \boldsymbol{W}\boldsymbol{n}, \boldsymbol{n}\times\boldsymbol{W}\boldsymbol{n}, \mathring{\boldsymbol{A}}\boldsymbol{n}, \boldsymbol{n}\times\mathring{\boldsymbol{A}}\boldsymbol{n}, \boldsymbol{\eta}_m(\mathring{\boldsymbol{A}}), \boldsymbol{n}\times\boldsymbol{\eta}_m(\mathring{\boldsymbol{A}});$$

and

$$\mathrm{Skw}^0(\boldsymbol{W}); \mathrm{Skw}^3_m(\boldsymbol{A}), \boldsymbol{n}\wedge\mathring{\boldsymbol{u}}, \boldsymbol{n}\wedge(\boldsymbol{n}\times\mathring{\boldsymbol{u}});$$

and

$$\mathrm{Sym}^0_{2m+1}(\boldsymbol{W}); \mathrm{Sym}^3_m(\boldsymbol{A}), \mathring{\boldsymbol{u}}\otimes\mathring{\boldsymbol{u}}, \mathring{\boldsymbol{u}}\vee(\boldsymbol{n}\times\mathring{\boldsymbol{u}}), \boldsymbol{n}\vee\mathring{\boldsymbol{u}}, \boldsymbol{n}\vee(\boldsymbol{n}\times\mathring{\boldsymbol{u}});$$

where $\boldsymbol{u}, \boldsymbol{v} = \boldsymbol{u}_1, \dots, \boldsymbol{u}_a$; $\boldsymbol{W}, \boldsymbol{H} = \boldsymbol{W}_1, \dots, \boldsymbol{W}_b$; $\boldsymbol{A}, \boldsymbol{B} = \boldsymbol{A}_1, \dots, \boldsymbol{A}_c$; $\boldsymbol{u} \neq \boldsymbol{v}$, $\boldsymbol{W} \neq \boldsymbol{H}$, *provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables* $X \in \mathcal{D}$ *under the group* $C_{2m+1}(\boldsymbol{n})$ *for each integer* $m \geqslant 1$, *respectively.*

8. Concluding remarks

In the previous sections, complete non-polynomial representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of any finite number of vectors and second-order tensors under all kinds of finite subgroups of the transverse isotropy group $C_{\infty h}$ are derived, each of which is composed of polynomial invariants or polynomial generators. The results presented supply unified forms of general representations for all subgroup classes concerned respectively. Of them, the results for generating sets are proved to be irreducible. Moreover, it is shown that each presented invariant with a single variable $\boldsymbol{u}$ or $\boldsymbol{W}$ or $\boldsymbol{A}$ is irreducible. Irreducibility of each presented invariant with two or three variables will be proved elsewhere.

Comparing the existing results for polynomial representations in some related particular cases (see, for example, Smith & Rivlin 1964; Smith & Kiral 1969; Kiral & Smith 1974) with the corresponding results presented here, one can see that non-polynomial representations are in general more compact than polynomial representations, as has been noticed before by Wang (1970) and Boehler (1979, 1987).

In Xiao (1996*d*), irreducible functional bases of a single symmetric tensor under the 32 crystal classes are derived, which are shown to be more compact than the well-known results for minimal integrity bases (see Smith & Rivlin 1958; see also Green & Adkins 1960; Truesdell & Noll 1965; Spencer 1971; Smith 1994). Here, even more compact results for the crystal classes S_6, C_{4h} and C_{6h} are obtained. In fact, each of the presented irreducible functional bases $I^3_m(\boldsymbol{A})$ for $m = 1$ and $I^0_m(\boldsymbol{A})$ for $m = 2, 3$ for the three crystal classes just mentioned, is composed of eight invariants only, whereas the corresponding bases given in Xiao (1996*c*) are formed by 11, 10 and 9 invariants respectively. Furthermore, each of the former provides a unified form of functional bases for all crystal and quasi-crystal classes concerned, and they always include eight invariants only. Inspired by these results, irreducible functional bases of a single symmetric tensor under all kinds of orthogonal subgroups have been derived in Xiao (1998*a*), each of which is of unified form for all infinitely many crystal and quasi-crystal classes concerned and consists merely of not more than eight invariants, except for the cubic crystal class T_h and the icosahedral quasi-crystal class I_h.

Some invariants and generators in the results presented are expressed in terms of summations with respect to sets of unit vectors each of which constitutes an equipartition of the unit circle in the $\boldsymbol{n}$-plane. In a recent article (Xiao 1998*b*), new forms of these results and a simplified derivation and proofs for them are available in terms of some trigonometric functions associated with Tschebysheff polynomials of the first and second kinds.

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